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GLOBAL DYNAMICS ABOVE THE GROUND STATE ENERGY FOR THE FOCUSING NONLINEAR KLEIN-GORDON EQUATION

K. NAKANISHI AND W. SCHLAG

ABSTRACT. The analysis of global dynamics of nonlinear dispersive equations has a long history starting from small solutions. In this paper we study the focusing, cubic, nonlinear Klein-Gordon equation in \mathbb{R}^3 with large radial data in the energy space. This equation admits a unique positive stationary solution Q , called the ground state. In 1975 Payne and Sattinger showed that solutions $u(t)$ with energy $E[u, \dot{u}]$ *strictly below* that of the ground state are divided into two classes, depending on a suitable functional $K(u)$: If $K(u) < 0$, then one has finite time blow-up, if $K(u) \geq 0$ global existence; moreover, these sets are invariant under the flow. Recently, Ibrahim, Masmoudi and the first author [22] improved this result by establishing scattering to zero for $K[u] \geq 0$ by means of a variant of the Kenig-Merle method [25], [26]. In this paper we go slightly beyond the ground state energy and we give a complete description of the evolution in that case. For example, in a small neighborhood of Q one encounters the following trichotomy: on one side of a center-stable manifold one has finite-time blow-up for $t \geq 0$, on the other side scattering to zero, and on the manifold itself one has scattering to Q , both as $t \rightarrow +\infty$. In total, the class of data with energy at most slightly above that of Q is divided into nine disjoint nonempty sets each displaying different asymptotic behavior as $t \rightarrow \pm\infty$, which includes solutions blowing up in one time direction and scattering to zero on the other. The analogue of the solutions found by Duyckaerts, Merle [13], [14] for the energy critical wave and Schrödinger equations appear here as the unique one-dimensional stable/unstable manifolds approaching $\pm Q$ exponentially as $t \rightarrow \infty$ or $t \rightarrow -\infty$, respectively. The main technical ingredient in our proof is a “one-pass” theorem which excludes the existence of (almost) homoclinic orbits between Q (as well as $-Q$) and (almost) heteroclinic orbits connecting Q with $-Q$. In a companion paper [31] we establish analogous properties for the NLS equation.

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1. INTRODUCTION

In this paper we study the global behavior of general solutions to the nonlinear Klein-Gordon equation (NLKG) with the focusing cubic nonlinearity on \mathbb{R}^3 , i.e.,

$$\ddot{u} - \Delta u + u = u^3, \quad u(t, x) : \mathbb{R}^{1+3} \rightarrow \mathbb{R}, \quad (1.1)$$

which conserves the energy

$$E(\vec{u}) := \int_{\mathbb{R}^3} \left[\frac{|\dot{u}|^2 + |\nabla u|^2 + |u|^2}{2} - \frac{|u|^4}{4} \right] dx. \quad (1.2)$$

We regard $H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ as the phase space for this infinite dimensional Hamiltonian system. In other words, we write the solutions as

$$\vec{u}(t) := (u(t), \dot{u}(t)) \in \mathcal{H} := H_{\text{rad}}^1 \times L_{\text{rad}}^2. \quad (1.3)$$

There exists a vast literature on the wellposedness theory for this equation in the energy space since Jörgens [23] and Segal [35], as well as the scattering theory for small data for the focusing nonlinearity as in (1.1) and for large data for the defocusing equation; see Brenner [7], [8], Ginibre, Velo [16], [17], Morawetz, Strauss [30], and Pecher [33]. In this paper, the scattering of a solution u to a static state φ refers to the following asymptotic behavior: there exists a solution v of the free Klein-Gordon equation such that

$$\|\vec{u}(t) - \vec{\varphi} - \vec{v}(t)\|_{\mathcal{H}} \rightarrow 0, \quad (1.4)$$

as either $t \rightarrow \infty$ or $t \rightarrow -\infty$ (depending on the context). See also Strauss [37] and [22] for a review of Strichartz estimates and wellposedness, as well as scattering in this setting.

It is well-known [36], [5], [11] that there exists a unique radial positive ground state $Q(x)$, solving the static equation

$$-\Delta Q + Q = Q^3, \quad (1.5)$$

with the least energy

$$E(Q) = J(Q) := \int_{\mathbb{R}^3} \left[\frac{|\nabla Q|^2 + |Q|^2}{2} - \frac{|Q|^4}{4} \right] dx > 0, \quad (1.6)$$

among the static solutions, and that the solutions u below the ground energy

$$E(\vec{u}) < E(Q) \quad (1.7)$$

are split into two classes by the functional

$$K_0(u) := \int_{\mathbb{R}^3} [|\nabla u|^2 + |u|^2 - |u|^4] dx = \partial_\lambda|_{\lambda=1} J(\lambda u), \quad (1.8)$$

- (1) If $K_0(u(0)) \geq 0$ then the solution u exists globally on $t \in \mathbb{R}$.
- (2) If $K_0(u(0)) < 0$ then the solution u blows up both in $\pm t > 0$.

In this paper, blow-up of a solution u means that it cannot be extended beyond a finite time T^* in the energy space \mathcal{H} . Then the wellposedness theory implies that $\|\vec{u}(t)\|_{\mathcal{H}} \rightarrow \infty$ as t approaches T^* .

The above result of dichotomy is essentially due to Payne, Sattinger [32], which of course implies that the ground state Q is unstable. For a general theory of orbital stability vs. instability of solitary wave solutions for equations of this type, see Grillakis, Shatah, Strauss [18], [19].

Recently, Kenig, Merle [25, 26] considered the focusing nonlinear Schrödinger equation (NLS) and the nonlinear wave equations with the energy-critical nonlinearity in three dimensions (as well as others) and obtained a stronger version of the dichotomy, by adding scattering to zero in the global existence scenario. It has been extended to several similar equations, including the cubic focusing NLS equation by Holmer, Roudenko [21], and NLKG such as (1.1) in [22].

On the other hand, the second author [34], followed by Beceanu [4], constructed center-stable manifolds with finite codimensions around the ground state in the case of the focusing cubic NLS equation in three dimensions. A construction of such manifolds for all L^2 supercritical NLS equations in one dimension was given by Krieger and the second author [27], and for the radial critical wave equation in \mathbb{R}^3 this was done in [28]. Stable, unstable and center manifolds have been known to play an important role in hyperbolic dynamics for a long time, see the classical work by Hirsch, Pugh, Shub [20] as well as many other works since then, such as Bates, Jones [2], [3], Vanderbauwhede [38], and Li, Wiggins [29]. While the more ODE oriented approach in Bates, Jones to PDEs such as (1.1) does construct center-stable and unstable local manifolds near an unstable equilibrium in the energy space, no statement can be made about the trajectories once they leave a small neighborhood of the equilibrium. In other words, they are local-in-time results which do not involve any dispersive analysis. In contrast, the emphasis in [34], [27], [28], [4] lies with the asymptotic behavior of the solutions as $t \rightarrow \infty$ and scattering to a suitable “soliton” is established in each of these works for data lying on a center-stable manifold modulo the group actions which preserve the equation. In [4] both points of view are united by establishing the existence of the manifold \mathcal{M} from [34] in the energy class (in fact, in the critical $\dot{H}^{\frac{1}{2}}$ topology) for the cubic NLS equation, with the added feature of global-in-time invariance of \mathcal{M} as $t \rightarrow \infty$, as well as scattering in the energy class to Q modulo the symmetries for solutions starting on \mathcal{M} .

Note that those solutions scattering to Q must have energy above the ground state, and so the solution sets of Payne, Sattinger on the one hand, and those on the center-stable manifolds on the other hand, are necessarily disjoint from each other. More recently, Duyckaerts, Merle [13, 14] investigated the solutions on the threshold energy $E(\vec{u}) = E(Q)$ for energy critical equations, and proved that there are exactly two new solutions modulo symmetry; scattering to Q as $t \rightarrow -\infty$, while either scattering to 0 or blowing up as $t \rightarrow \infty$. They can be regarded as minimal

solutions on the manifold \mathcal{M} . This result is also extended to the cubic NLS by Duyckaerts, Roudenko [15]. In this paper we exhibit these solutions as the unique stable/unstable manifolds associated with $\pm Q$, see [3] for the definition of these objects.

However, all of these works describe only part of the dynamics for energies near that of the ground state. A natural question to ask is whether the solutions near the soliton are separated by \mathcal{M} into a region of scattering to zero, and the other region of blow-up. This is partially motivated by the study of “critical phenomena” in the physics literature, see for example Choptuik [9], Choptuik, Chmaj, Bizon [10], and Bizon, Chmaj, Tabor [6]. For the $|u|^5$ wave equation in \mathbb{R}^3 , blow-up for all data on one side of the tangent space to the center-stable manifold constructed in [28] was shown by Karageorgis, Strauss [24].

Our goal in this paper is to give a complete picture of the dynamics of (1.1) for radial energy data with energy slightly larger than that of Q :

$$\mathcal{H}^\varepsilon := \{\vec{u} \in \mathcal{H} \mid E(\vec{u}) < E(Q) + \varepsilon^2\}. \quad (1.9)$$

Note that the only symmetry in \mathcal{H} is $u \mapsto -u$ in this setting.

Theorem 1.1. *Consider all solutions of NLKG (1.1) with initial data $\vec{u}(0) \in \mathcal{H}^\varepsilon$ for some small $\varepsilon > 0$. We prove that the solution set is decomposed into nine non-empty sets characterized as*

- (1) *Scattering to 0 for both $t \rightarrow \pm\infty$,*
- (2) *Finite time blow-up on both sides $\pm t > 0$,*
- (3) *Scattering to 0 as $t \rightarrow \infty$ and finite time blow-up in $t < 0$,*
- (4) *Finite time blow-up in $t > 0$ and scattering to 0 as $t \rightarrow -\infty$,*
- (5) *Trapped by $\pm Q$ for $t \rightarrow \infty$ and scattering to 0 as $t \rightarrow -\infty$,*
- (6) *Scattering to 0 as $t \rightarrow \infty$ and trapped by $\pm Q$ as $t \rightarrow -\infty$,*
- (7) *Trapped by $\pm Q$ for $t \rightarrow \infty$ and finite time blow-up in $t < 0$,*
- (8) *Finite time blow-up in $t > 0$ and trapped by $\pm Q$ as $t \rightarrow -\infty$,*
- (9) *Trapped by $\pm Q$ as $t \rightarrow \pm\infty$,*

where “trapped by $\pm Q$ ” means that the solution stays in a $O(\varepsilon)$ neighborhood of $\pm Q$ forever after some time (or before some time). The initial data sets for (1)-(4), respectively, are open.

The striking difference from the Kenig-Merle or Duyckaerts-Merle type results is the existence of solutions which blow up for $t < 0$ and scatter for $t \rightarrow +\infty$, or vice versa. It also implies that the initial data set for the forward scattering (1) \cup (3) \cup (6) (or backward scattering) is unbounded in \mathcal{H} ; in fact, it contains a curve connecting zero to infinity in \mathcal{H} . The number “nine” simply means that all possible combinations of scattering to zero/scattering to $\pm Q$ /finite time blow-up are allowed as $t \rightarrow \pm\infty$. Each of these are in fact realized by infinitely many solutions.

The simple dichotomy in terms of K_0 is no longer available, so it is not easy to predict the global dynamics for a given initial data, as in the case of Payne-Sattinger or Kenig-Merle below the ground state. However, we can give some description, in a more dynamical way, combining the hyperbolic structure around the ground state with the global variational structure. Let L_+ denote the linearized operator around

Q , and let ρ be its ground state:

$$L_+ = -\Delta + 1 - 3Q^2, \quad L_+\rho = -k^2\rho, \quad k > 0, \quad \rho(x) > 0, \quad \|\rho\|_{L^2} = 1. \quad (1.10)$$

It is well known that L_+ has only one negative eigenvalue. We can define a nonlinear distance function $d_Q(\vec{u}) : \mathcal{H}^\varepsilon \rightarrow [0, \infty)$ continuous such that

$$\begin{aligned} d_Q(\vec{u}) &\simeq \inf_{\pm} \|\vec{u} \mp \vec{Q}\|_{\mathcal{H}}, \\ d_Q(\vec{u}) \ll 1 &\implies d_Q^2(\vec{u}) = E(\vec{u}) - E(Q) + k^2|\langle u \mp Q | \rho \rangle|^2, \end{aligned} \quad (1.11)$$

where $\pm Q$ is chosen to be the closest to u . Let

$$B(\pm Q) := \{\vec{u} \in \mathcal{H}^\varepsilon \mid d_Q^2(\vec{u}) \leq 2[E(\vec{u}) - E(Q)]\} \quad (1.12)$$

be a pair of small balls around $\pm Q$. Outside of these balls, we can define a continuous sign function $\mathfrak{S} : \mathcal{H}^\varepsilon \setminus B(\pm Q) \rightarrow \{\pm 1\}$ such that

$$\begin{aligned} d_Q(\vec{u}) \leq \delta_X &\implies \mathfrak{S}(\vec{u}) = -\text{sign}\langle u \mp Q | \rho \rangle, \\ d_Q(\vec{u}) \geq \delta_S &\implies \mathfrak{S}(\vec{u}) = \text{sign}K_0(u), \end{aligned} \quad (1.13)$$

for some $\delta_X > \delta_S > 2\varepsilon$, with the convention that $\text{sign}0 = +1$. We emphasize that \mathfrak{S} is different¹ from $\text{sign}K_0$ in the region close to $\pm Q$.

Theorem 1.2. *There are small $\varepsilon > 0$ and $R > 2\varepsilon$ with the following property. If u is a solution of NLKG (1.1) in \mathcal{H}^ε , defined on an interval I , satisfying*

$$d_Q(\vec{u}(\tau_1)) < d_Q(\vec{u}(\tau_2)) < R, \quad \vec{u}(\tau_2) \notin B(\pm Q), \quad (1.14)$$

for some $\tau_1 < \tau_2 \in I$, then we have $d_Q(\vec{u}(t)) > d_Q(\vec{u}(\tau_2))$ for all $t > \tau_2$ in I . In particular, u remains outside of $B(\pm Q)$ with a fixed sign $\mathfrak{S}(\vec{u}(t)) \in \{\pm 1\}$ for $t > \tau_2$. If $\mathfrak{S}(\vec{u}(t)) = +1$, then u scatters to 0 as $t \rightarrow \infty$. If $\mathfrak{S}(\vec{u}(t)) = -1$, then u blows up in finite time after τ_2 . Conversely, if a solution u in \mathcal{H}^ε scatters as $t \rightarrow \infty$, then $d_Q(\vec{u}(t)) > R$, $\vec{u}(t) \notin B(\pm Q)$ and $\mathfrak{S}(\vec{u}(t)) = +1$ for large t . If it blows up as $t \rightarrow T - 0$, then $d_Q(\vec{u}(t)) > R$, $\vec{u}(t) \notin B(\pm Q)$ and $\mathfrak{S}(\vec{u}(t)) = -1$ for $t < T$ close to T . If it is trapped by $\pm Q$ as $t \rightarrow \infty$, then it stays in $B(\pm Q)$ for large t .

In other words, every solution can enter and exit $B(\pm Q)$ at most once, and the sign of $\mathfrak{S}(\vec{u})$ is constant while it is away from $B(\pm Q)$. Hence, if a solution does enter $B(\pm Q)$ and exits, then its fate is determined by the sign of \mathfrak{S} at the entrance and the exit, respectively, for $t \rightarrow -\infty$ and $t \rightarrow +\infty$, which are not necessarily the same (cf. cases (3) and (4) of Theorem 1.1). It also implies that the set (9) in the previous theorem is bounded in the energy space \mathcal{H} .

In fact, we can give a more precise description of the exiting dynamics. Indeed, in the setting (1.14), $d_Q(\vec{u}(t))$ is monotonically and exponentially growing for $t \geq \tau_2$ until it reaches a larger number δ_X (see Lemma 4.2).

One can also give a more detailed description of the dynamics of those solutions which are trapped by $\pm Q$, by means of the following *spectral gap property* of the

¹ \mathfrak{S} might be the same as $\text{sign}K_0$ in $\mathcal{H}^\varepsilon \setminus B(\pm Q)$, but it depends on the coefficient 2 in the definition of $B(\pm Q)$. In fact we can make it closer to 1 as $d_Q(\vec{u}) \rightarrow 0$, and in that region we can show that $-\text{sign}\langle u \mp Q | \rho \rangle$ and $\text{sign}K_0(u)$ are indeed opposite at some points.

linearized operator $L_+ = -\Delta + 1 - 3Q^2$:

$$L_+ \text{ has no eigenvalues in } (0, 1] \text{ and no resonance}^2 \text{ at the threshold } 1 \quad (1.15)$$

It was verified by means of the Birman-Schwinger theorem and a numerical computation of the five largest eigenvalues of a suitably discretized Birman-Schwinger operator in [12] by Demanet and the second author. Using this gap property one obtains the following refinement of Theorem 1.1.

Theorem 1.3. *In the statement of Theorem 1.1 one can replace “trapped by $\pm Q$ ” with “scattering to $\pm Q$ ”. The sets $(5) \cup (7) \cup (9)$ and $(6) \cup (8) \cup (9)$ are smooth codimension one manifolds in the (radial) phase space \mathcal{H} , and they are the³ center-stable manifold, resp. the center-unstable manifold, around $\pm Q$. Similarly, (9) is a smooth manifold of codimension 2, namely the center manifold.*

It seems natural to the authors to separate the above theorem from the previous ones, as it is known from [12] that (1.15) fails if one lowers the power 3 on the nonlinearity slightly, say to power < 2.8 . On the other hand, our argument for Theorems 1.1 and 1.2 is quite general, in particular it does not use (1.15), or more precisely, any dispersive property of the linearized operator. In fact, it is straightforward to extend them to all L^2 super-critical and H^1 subcritical powers and all space dimensions, i.e.

$$\ddot{u} - \Delta u + u = u^p, \quad 1 + 4/d < p < 1 + 4/(d - 2), \quad u : \mathbb{R}^{1+d} \rightarrow \mathbb{R}. \quad (1.16)$$

Finally, from the above theorems, we can easily deduce a Duyckaerts-Merle-type result on the energy threshold.

Theorem 1.4. *Consider the limiting case $\varepsilon \rightarrow 0$ in Theorem 1.1, i.e., all the radial solutions satisfying $E(\vec{u}) \leq E(Q)$. Then the sets (3) and (4) vanish, while the sets (5) – (9) are characterized, with some special solutions W_\pm , as follows*

$$\begin{aligned} (5) &= \{\pm W_-(t - t_0) \mid t_0 \in \mathbb{R}\}, & (6) &= \{\pm W_-(-t - t_0) \mid t_0 \in \mathbb{R}\}, \\ (7) &= \{\pm W_+(t - t_0) \mid t_0 \in \mathbb{R}\}, & (8) &= \{\pm W_+(-t - t_0) \mid t_0 \in \mathbb{R}\}, \\ (9) &= \{\pm Q(t - t_0) \mid t_0 \in \mathbb{R}\}. \end{aligned} \quad (1.17)$$

The solutions $W_\pm(t)$ converge exponentially to Q as $t \rightarrow \infty$.

As mentioned before $(5) \cup (7) \cup (9)$ is the stable manifold, and $(6) \cup (8) \cup (9)$ the unstable manifold, associated with $\pm Q$, cf. [3]. The rest of the paper is organized as follows. In Section 2, we recall several known facts about the ground state and the linearized operator around it. In Section 3 we construct the center-stable manifold around the ground states, under the aforementioned spectral condition (1.15). Next to Theorem 1.3 this is the *only part* where we require the gap property. Section 4 is the most important and novel part of this paper, where we prove a part of Theorem 1.2 that every solution can enter and exit a small neighborhood of the ground states

²In our radial case, it is easy to preclude by means of analytical arguments eigenvalues at the threshold 1, but we include it here, because such an eigenvalue might exist in the nonradial case, which is indeed treated by [12].

³Since center manifolds are in general not unique it might be more precise to say “a center-stable manifold” here, but we ignore this issue. In fact, our manifolds are naturally unique for the global characterization in Theorem 1.1.

$\pm Q$ at most one time. We call it “one-pass theorem”. One then immediately obtains the blow-up part of the dynamical classification by the classical Payne, Sattinger argument. In Section 5, we then prove the scattering part, by complementing the Kenig, Merle argument with the one-pass theorem. Finally in Section 6, we describe the global dynamics and its classification as in Theorem 1.1 and Theorem 1.3, using some simple topological arguments and the one-pass theorem. In the appendix we give a table of notation for frequently used symbols.

2. THE GROUND STATE

Here we recall several known properties of the ground state Q . First we consider its variational character with respect to the scaling symmetry. For our purpose, it suffices to consider the L^0 and the L^2 invariant scalings:

$$\varphi(x) \mapsto \varphi_0^\nu(x) := \nu\varphi(x), \quad \varphi(x) \mapsto \varphi_2^\nu(x) := \nu^{3/2}\varphi(\nu x). \quad (2.1)$$

Let D_0 and D_2 , respectively, be the generators of these symmetries, viz.

$$D_0\varphi(x) := \varphi(x) = \partial_\nu|_{\nu=1}\varphi_0^\nu, \quad D_2\varphi(x) := \frac{x\nabla + \nabla x}{2}\varphi(x) = \partial_\nu|_{\nu=1}\varphi_2^\nu, \quad (2.2)$$

and let K_0 and K_2 be the derivatives of J with respect to these scalings:

$$\begin{aligned} K_0(\varphi) &:= \int_{\mathbb{R}^3} [|\nabla\varphi|^2 + |\varphi|^2 - |\varphi|^4] dx = \partial_\nu|_{\nu=1}J(\varphi_0^\nu) = \langle J'(\varphi) | D_0\varphi \rangle, \\ K_2(\varphi) &:= \int_{\mathbb{R}^3} [|\nabla\varphi|^2 - \frac{3}{4}|\varphi|^4] dx = \partial_\nu|_{\nu=1}J(\varphi_2^\nu) = \langle J'(\varphi) | D_2\varphi \rangle, \end{aligned} \quad (2.3)$$

where $J'(\varphi)$ denotes the Fréchet derivative

$$J'(\varphi) = -\Delta\varphi + \varphi - \varphi^3. \quad (2.4)$$

Define positive functionals G_s for $s = 0, 2$ by

$$G_0(\varphi) := J - \frac{K_0}{4} = \frac{1}{4}\|\varphi\|_{H^1}^2, \quad G_2(\varphi) := J - \frac{K_2}{3} = \frac{1}{6}\|\nabla\varphi\|_{L^2}^2 + \frac{1}{2}\|\varphi\|_{L^2}^2. \quad (2.5)$$

Lemma 2.1. *For $s = 0, 2$ we have*

$$\begin{aligned} J(Q) &= \inf\{J(\varphi) \mid 0 \neq \varphi \in H^1, K_s(\varphi) = 0\} \\ &= \inf\{G_s(\varphi) \mid 0 \neq \varphi \in H^1, K_s(\varphi) \leq 0\}, \end{aligned} \quad (2.6)$$

and these infima are achieved uniquely by the ground states $\pm Q$.

Proof. If $K_s(\varphi) < 0$ then $K_s(\lambda_*\varphi) = 0$ for some $\lambda_* \in (0, 1)$, whereas $G_s(\lambda_*\varphi) < G_s(\varphi)$. Since moreover $K_s(\varphi) = 0$ implies $J(\varphi) = G_s(\varphi)$, the two infima are equal. Since $J'(Q) = 0$ implies $K_s(Q) = 0$, the infima are no larger than $J(Q)$.

To obtain a minimizer, let $\{\varphi_n\}_{n \geq 1} \subset H_{\text{rad}}^1 \setminus \{0\}$ be a minimizing sequence such that (the Schwartz symmetrization allows us to restrict them to the radial functions)

$$K_s(\varphi_n) = 0, \quad J(\varphi_n) \rightarrow m, \quad (2.7)$$

where m denotes the right hand side of (2.6). Since $G_s(\varphi_n) = J(\varphi_n)$ is bounded, φ_n is bounded in H^1 . After extraction of a subsequence, it converges weakly to some φ_∞ in H^1 , and in the strong sense in L^4 , by the radial symmetry. Thus $K_s(\varphi_\infty) \leq 0$, $J(\varphi_\infty) \leq J(Q)$ and $G_s(\varphi_\infty) \leq G_s(Q)$. If $\varphi_\infty = 0$ then the strong convergence in

L^4 together with $K_s(\varphi_n) \rightarrow 0$ implies that $\varphi_n \rightarrow 0$ strongly in H^1 . Notice that $K_s(\varphi) > 0$ for $0 < \|\varphi\|_{H^1} \ll 1$, due to the interpolation inequality

$$\|\varphi\|_{L^4}^4 \lesssim \|\varphi\|_{L^2} \|\nabla \varphi\|_{L^2}^3. \quad (2.8)$$

Hence $K_s(\varphi_n) > 0$ for large n , a contradiction. Thus we obtain a nonzero minimizer φ_∞ , and so $K_s(\varphi_\infty) = 0$, which implies that $\varphi_n \rightarrow \varphi_\infty$ strongly in H^1 .

The constrained minimization implies that for some Lagrange multiplier $\mu \in \mathbb{R}$,

$$J'(\varphi_\infty) = \mu K'_s(\varphi_\infty), \quad K_s(\varphi_\infty) = 0, \quad (2.9)$$

Multiplying the Euler-Lagrange equation with $D_s \varphi$ and integrating by parts yields

$$0 = K_s(\varphi_\infty) = \mu \langle K'_s(\varphi_\infty) | D_s(\varphi_\infty) \rangle = \mu \times \begin{cases} 2K_0(\varphi_\infty) - 2\|\varphi_\infty\|_{L^4}^4 & (s=0), \\ 2K_2(\varphi_\infty) - \frac{3}{4}\|\varphi_\infty\|_{L^4}^4 & (s=2), \end{cases} \quad (2.10)$$

which implies $\mu = 0$, so φ_∞ is a ground state. \square

Next, we record the following observation from [22].

Lemma 2.2. *For any $\varphi \in H^1$ such that $J(\varphi) < J(Q)$, one has either $\varphi = 0$, or $K_s(\varphi) > 0$ for both $s = 0, 2$, or $K_s(\varphi) < 0$ for both $s = 0, 2$.*

Proof. In the proof of the above lemma, we already know that 0 is surrounded by the open set $K_s(\varphi) > 0$ for both s , and $K_s(\varphi) = 0$ is prohibited except for 0 by $J(\varphi) < J(Q)$, due to the above minimization property of Q . Then it is enough to show that each set $\mathcal{K}_s^+ := \{K_s \geq 0, J < J(Q)\}$ is contractible to $\{0\}$, which implies that it is connected and so cannot be divided by the disjoint open sets \mathcal{K}_{2-s}^+ and $\mathcal{K}_{2-s}^- = \{K_{2-s} < 0, J < J(Q)\}$. The contraction is given by φ_s^ν with $\nu : 1 \rightarrow 0$. By definition $\partial_\nu J(\varphi_s^\nu) = K_s(\varphi_s^\nu)/\nu$, and so $J(\varphi_s^\nu)$ decreases together with ν , as long as $K_s(\varphi_s^\nu) > 0$, which is preserved as long as $J(\varphi_s^\nu) < J(Q)$. Hence, φ_s^ν stays in \mathcal{K}_s^+ for $1 \geq \nu > 0$. When $s = 2$, it converges to 0 as $\nu \rightarrow +0$ only in $\dot{H}^1 \cap L^4$, but since \mathcal{K}_2^\pm are open in that topology this is sufficient: once φ_2^ν gets in a small ball around 0 in \mathcal{K}_2^+ , one can change to the other scaling $\varphi \mapsto \nu \varphi$ to send it to 0 in H^1 . \square

Next we recall the spectral properties of Q and the linearized operator L_+ defined in (1.10). Decomposing the solution of (1.1) in the form

$$u = Q + v, \quad (2.11)$$

we obtain the equation of the remainder

$$\ddot{v} + L_+ v = N(v), \quad N(v) := (v + Q)^3 - Q^3 - 3Q^2 v = 3Qv^2 + v^3. \quad (2.12)$$

The energy functionals are expanded correspondingly

$$\begin{aligned} J(Q + v) &= J(Q) + \langle J'(Q) | v \rangle + \langle J''(Q) v | v \rangle / 2 + O(\|v\|_{H^1}^3) \\ &= J(Q) + \langle L_+ v | v \rangle / 2 + O(\|v\|_{H^1}^3), \\ K_s(Q + v) &= \langle K'_s(Q) | v \rangle + \langle K''_s(Q) v | v \rangle / 2 + O(\|v\|_{H^1}^3). \end{aligned} \quad (2.13)$$

In particular

$$\begin{aligned} K_0(Q + v) &= 2\langle -\Delta Q + Q - 2Q^3 | v \rangle + \langle (-\Delta + 1 - 6Q^2) v | v \rangle + O(\|v\|_{H^1}^3) \\ &= -2\langle Q^3 | v \rangle + \langle (L_+ - 3Q^2) v | v \rangle + O(\|v\|_{H^1}^3). \end{aligned} \quad (2.14)$$

Lemma 2.3. *As an operator in L^2_{rad} , L_+ has only one negative eigenvalue, which is non-degenerate, and no eigenvalue at 0 or in the continuous spectrum $[1, \infty)$.*

Proof. L_+ has at least one negative eigenvalue because

$$\langle L_+ Q | Q \rangle = -3 \|Q\|_4^4 < 0, \quad (2.15)$$

and at most one, because

$$\langle Q^3 | v \rangle = 0 \implies \langle L_+ v | v \rangle \geq 0. \quad (2.16)$$

To see this, suppose that $f \in H^1$ satisfies $\langle Q^3 | f \rangle = 0$ and $\langle L_+ f | f \rangle = -1$, and let $v = \varepsilon Q + \delta f$ for small $\varepsilon, \delta \in \mathbb{R}$. Then

$$K_0(Q + v) = -2\varepsilon \|Q\|_{L^4}^4 - \delta^2(1 + \langle 3Q^2 f | f \rangle) + O(\varepsilon^2 + \varepsilon\delta + \delta^3), \quad (2.17)$$

so there exists $\varepsilon = O(\delta^2)$ such that $K_0(Q + v) = 0$. On the other hand

$$J(Q + v) = J(Q) - \delta^2 + O(\delta^3) < J(Q), \quad (2.18)$$

which contradicts the minimizing property Lemma 2.1.

Next, if $0 \neq f \in L^2_{\text{rad}}$ solves $L_+ f = 0$, then $f \perp Q^3, Q$, because $L_+ Q = -2Q^3$ and $L_+(r\partial_r + 1)Q = -2Q$. By the Sturm-Liouville theory, f can change the sign only once, say at $r = r_0 > 0$, so does $Q^3 - Q(r_0)^2 Q$, since $Q(r)$ is decreasing. Then $(Q^3 - Q(r_0)^2 Q)f$ is non-zero with a definite sign, contradicting $f \perp Q^3, Q$.

The absence of embedded eigenvalue is standard, and follows also from the asymptotic equation for any eigenfunction, viz.

$$L_+ f = \lambda^2 f \implies (\partial_r^2 + \lambda^2)(rf) = -3Q^2(rf) \lesssim e^{-2r}, \quad (2.19)$$

by the exponential decay of Q . \square

Let

$$P^+ := 1 - \rho \langle \rho | \quad (2.20)$$

be the orthogonal projection for ρ . Then the above lemma implies $\langle L_+ v | P^+ v \rangle \gtrsim \|v\|_{L^2}^2$, and so for any $v \in P^+(H^1)$ and $\theta \in (0, 1]$ sufficiently small,

$$\langle L_+ v | v \rangle \gtrsim (1 - \theta) \|v\|_2^2 + \theta \|v\|_{H^1}^2 - 3\theta \langle Q^2 v | v \rangle \gtrsim \|v\|_{H^1}^2, \quad (2.21)$$

hence $\langle L_+ v | v \rangle \simeq \|v\|_{H^1}^2$ on $P^+(H^1)$.

The above property of L_+ is sufficient for the analysis of dynamics away from Q , but for that of solutions staying forever around the ground state, we require the following property of L_+ :

$$L_+ \text{ has no eigenvalue in } (0, 1] \text{ and no resonance at the threshold } 1. \quad (2.22)$$

Both parts of this statement have been verified by Demanet and the second author in [12] via numerics. Their approach is based on the Birman-Schwinger theorem which equates the number of eigenvalues and resonances ≤ 1 (the threshold) of L_+ , counted with multiplicity, to the number of eigenvalues ≥ 1 of the self-adjoint, positive, compact operator $K_+ := 3Q(-\Delta)^{-1}Q$, again counted with multiplicity. In fact, [12] by means of a numerical computation finds that K_+ has precisely four eigenvalues greater than 1 (corresponding to the negative ground state plus the zero eigenvalue of multiplicity three: $L_+ \nabla Q = 0$), whereas the fifth largest eigenvalue

was calculated to be < 0.98 with an estimated 8 to 9 digits of accuracy behind the comma (to be precise, $\lambda_5 = 0.97039244\dots$). By the Birman-Schwinger theorem, this verifies (2.22).

3. CENTER-STABLE MANIFOLD

In this section, we investigate the solutions staying around the unstable ground state, constructing the center-stable manifold, while the center-unstable manifold is obtained by reversing the time. The situation is much simpler than for NLS [34], because the only symmetry present (reflection and time translation) fixes the ground state and the linearized operator is scalar self-adjoint. We decompose any solution u simply by putting

$$u(t) = Q + v(t), \quad v(t) = \lambda(t)\rho + \gamma, \quad \gamma \perp \rho, \quad (3.1)$$

which is obviously unique. We then obtain the equations of $(\lambda, \gamma) \in \mathbb{R} \times P^+(H^1)$

$$\begin{cases} \ddot{\lambda} - k^2\lambda = P_\rho N(v) =: N_\rho(v), \\ \ddot{\gamma} + \omega^2\gamma = P^+ N(v) =: N_c(v), \quad \omega := \sqrt{P^+ L_+}. \end{cases} \quad (3.2)$$

We look for a forward global solution which grows at most polynomially, to which it is equivalent to remove the growing mode e^{kt} . From the integral equation of λ one extracts the growing mode

$$\begin{aligned} \lambda(t) &= \cosh(kt)\lambda(0) + \frac{1}{k}\sinh(kt)\dot{\lambda}(0) + \frac{1}{k}\int_0^t \sinh(k(t-s))N_\rho(v)(s)ds \\ &= \frac{e^{kt}}{2} \left[\lambda(0) + \frac{1}{k}\dot{\lambda}(0) + \frac{1}{k}\int_0^t e^{-ks}N_\rho(v)(s)ds \right] + \dots \end{aligned} \quad (3.3)$$

where the omitted terms are exponentially decaying. Hence, the necessary and sufficient stability condition is

$$\dot{\lambda}(0) = -k\lambda(0) - \int_0^\infty e^{-ks}N_\rho(v)(s)ds. \quad (3.4)$$

Under this condition, the integral equation for $\lambda(t)$ is reduced to

$$\lambda(t) = e^{-kt} \left[\lambda(0) + \frac{1}{2k}\int_0^\infty e^{-ks}N_\rho(v)(s)ds \right] + \frac{1}{2k}\int_0^\infty e^{-k|t-s|}N_\rho(v)(s)ds, \quad (3.5)$$

while the integral equation for γ is

$$\gamma(t) = \cos(\omega t)\gamma(0) + \frac{1}{\omega}\sin(\omega t)\dot{\gamma}(0) + \frac{1}{\omega}\int_0^t \sin(\omega(t-s))N_c(v)(s)ds. \quad (3.6)$$

The coupled equations (3.5)–(3.6) can be solved by iteration, using the Strichartz estimate for $e^{it\omega}$, for any small initial data $(\lambda(0), \dot{\gamma}(0)) \in \mathbb{R} \times P^+(\mathcal{H})$. The linearized energy norm of v around Q is denoted by

$$\begin{aligned} \|\vec{v}\|_E &:= \sqrt{[k^2\langle v|\rho\rangle^2 + \|\omega P^+v\|_{L^2}^2 + \|\dot{v}\|_{L^2}^2]/2} \\ &= \sqrt{[k^2|\lambda|^2 + |\dot{\lambda}|^2]/2 + \|\gamma\|_E^2}. \end{aligned} \quad (3.7)$$

By Lemma 2.3 we have

$$\|\vec{v}\|_E^2 \simeq \|\vec{v}\|_{\mathcal{H}}^2 = \|v\|_{H^1}^2 + \|\dot{v}\|_{L^2}^2. \quad (3.8)$$

Recall that \mathcal{H} is the radial energy space.

Proposition 3.1. *Assume that (2.22) holds. Then there are $\nu > 0$ and $C \geq 1$ with the following property: For any given $\lambda(0) \in \mathbb{R}$, $\vec{\gamma}(0) \in P^+(\mathcal{H})$ satisfying*

$$E_0 := k^2|\lambda(0)|^2 + \|\omega\gamma(0)\|_{L^2}^2 + \|\dot{\gamma}(0)\|_{L^2}^2 \leq \nu^2, \quad (3.9)$$

there exists a unique solution u of NLKG (1.1) on $0 \leq t < \infty$ satisfying

$$u(0) = Q + \lambda(0)\rho + \gamma(0), \quad P^+\dot{u}(0) = \dot{\gamma}(0), \quad (3.10)$$

$|\langle \rho | \dot{u}(t) + ku(t) \rangle| \lesssim E_0$ for all $t \geq 0$, and

$$\|\vec{u}(t) - \vec{Q}\|_{\mathcal{H}}^2 \leq CE_0 \quad (0 \leq \forall t < \infty). \quad (3.11)$$

The dependence of u on $(\lambda(0), \vec{\gamma}(0))$ is smooth in $L^\infty(0, \infty; \mathcal{H})$. In addition, there exists a unique free Klein-Gordon solution γ_∞ such that

$$|\lambda(t)| + |\dot{\lambda}(t)| + \|\vec{\gamma}(t) - \vec{\gamma}_\infty(t)\|_{\mathcal{H}} \rightarrow 0, \quad (3.12)$$

as $t \rightarrow \infty$. In particular, we have $E(\vec{u}) = J(Q) + \|\vec{\gamma}_\infty\|_{\mathcal{H}}^2/2$.

Conversely, any solution u of NLKG satisfying (3.11) with $E_0 \leq \nu^2/C$ must be given in this way, which is uniquely determined by $(\lambda(0), \vec{\gamma}(0))$, and by $(\lambda(0), \vec{\gamma}_\infty(0))$.

An alternate, but essentially equivalent, formulation of this result reads as follows.

Proposition 3.2. *Assume that (2.22) holds. Then there exists $\nu > 0$ small and a smooth graph \mathcal{M} in $B_\nu(Q) \subset \mathcal{H}$ so that \mathcal{M} is tangent to*

$$T_Q\mathcal{M} = \{(u_0, u_1) \in \mathcal{H} \mid \langle ku_0 + u_1 | \rho \rangle = 0\} \quad (3.13)$$

in the sense that

$$\sup_{x \in \partial B_\delta(Q)} \text{dist}(x, T_Q\mathcal{M}) \lesssim \delta^2 \quad \forall 0 < \delta < \nu$$

and so that any data $(u_0, u_1) \in \mathcal{M}$ lead to global evolutions of (1.1) of the form $u = Q + v$ where v scatters to a free Klein-Gordon solution in \mathcal{H} . Moreover, no solution can stay off \mathcal{M} and inside $B_\nu(Q)$ for all $t > 0$, and, \mathcal{M} is invariant under the flow for all $t \geq 0$.

Proof of Proposition 3.1. For the existence, we solve (3.5)–(3.6) by iteration using the norm

$$\|(\lambda, \gamma)\|_X := \|\lambda\|_{L^1 \cap L^\infty(0, \infty)} + \|\gamma\|_{St(0, \infty)}, \quad St := L_t^2 L_x^6 \cap L_t^\infty H_x^1. \quad (3.14)$$

The Strichartz estimate for the free Klein-Gordon equation gives us

$$\|u\|_{St(0, T)} \lesssim \|\vec{u}(0)\|_{\mathcal{H}} + \|(\partial_t^2 - \Delta + 1)u\|_{L_t^1 L_x^2(0, T)}. \quad (3.15)$$

Under the hypothesis (2.22), the operator L_+ satisfies the conditions of Yajima's $W^{k,p}$ boundedness theorem for the wave operators [39], so that we can conclude that (3.15) applies to γ as well:

$$\|\gamma\|_{St(0, T)} \lesssim \|\vec{\gamma}(0)\|_{\mathcal{H}} + \|(\partial_t^2 - \omega^2)\gamma\|_{L_t^1 L_x^2(0, T)}, \quad (3.16)$$

provided that $\vec{\gamma}(0)$ and $(\partial_t^2 - \omega^2)\gamma$ are orthogonal to ρ .

For the solution λ of (3.5), with $\dot{\lambda}(0)$ uniquely determined by (3.4), we can estimate the norm by simple integration in t :

$$\|\lambda\|_{L^1 \cap L^\infty(0,\infty)} \lesssim k^{-1}|\lambda(0)| + k^{-1}\|N_\rho(v)\|_{L_t^1(0,\infty)} \lesssim |\lambda(0)| + \|N(v)\|_{L_t^1 L_x^2(0,\infty)}, \quad (3.17)$$

and for the solution γ of (3.6) by the above Strichartz estimate

$$\|\gamma\|_{St(0,\infty)} \lesssim \|\vec{\gamma}(0)\|_{\mathcal{H}} + \|N(v)\|_{L_t^1 L_x^2(0,\infty)}. \quad (3.18)$$

The nonlinearity $N(v)$ is bounded in $L_t^1 L_x^2$ by

$$\|Qv^2\|_{L_t^1 L_x^2} + \|v^3\|_{L_t^1 L_x^2} \lesssim \|v\|_{L_t^2 L_x^6}^2 (\|v\|_{L_t^\infty L_x^6} + \|Q\|_{L_t^\infty L_x^6}), \quad (3.19)$$

where the norm of v is bounded by

$$\|v\|_{L_t^p L_x^6} \lesssim \|\lambda\|_{L_t^1 \cap L_t^\infty} + \|\gamma\|_{L_t^p L_x^6} \quad (1 \leq p \leq \infty). \quad (3.20)$$

Gathering them, we obtain

$$\|(\lambda, \gamma)\|_X \lesssim |\lambda(0)| + \|\vec{\gamma}(0)\|_{\mathcal{H}} + \|(\lambda, \gamma)\|_X^2 + \|(\lambda, \gamma)\|_X^3. \quad (3.21)$$

Applying these estimates to the iteration sequence, we obtain a unique fixed point of (3.5)–(3.6) for any given small $(\lambda(0), \vec{\gamma}(0))$. It is straightforward to see that $u := Q + \lambda(t)\rho + \gamma(t)$ solves NLKG on $0 \leq t < \infty$, satisfying

$$\|\vec{u} - \vec{Q}\|_{\mathcal{H}} \lesssim \|(\lambda, \gamma)\|_X \lesssim |\lambda(0)| + \|\vec{\gamma}(0)\|_{\mathcal{H}}, \quad (3.22)$$

with smooth dependence on the data. The bound on $\langle \rho | \dot{u} + ku \rangle = \dot{\lambda} + k\lambda$ follows by using the equation once again.

Moreover, the asymptotic profile of γ is given by

$$\gamma_\infty(t) = \cos(\omega t)\gamma(0) + \frac{1}{\omega} \sin(\omega t)\dot{\gamma}(0) + \frac{1}{\omega} \int_0^\infty \sin(\omega(t-s))N_c(v)(s) ds, \quad (3.23)$$

with the convergence property

$$\|\vec{\lambda}\|_{L_t^\infty(T,\infty)} + \|\vec{\gamma} - \vec{\gamma}_\infty\|_{L_t^\infty \mathcal{H}(T,\infty)} \lesssim \|N(v)\|_{L_t^1 L^2(T,\infty)} \rightarrow 0 \quad (T \rightarrow \infty). \quad (3.24)$$

The iteration can be solved with a given γ_∞ and the equation of γ now reads

$$\gamma(t) = \gamma_\infty(t) + \frac{1}{\omega} \int_\infty^t \sin(\omega(t-s))N_c(v)(s) ds, \quad (3.25)$$

where the estimates are essentially the same. $\gamma_\infty(t)$ can be further replaced with a free Klein-Gordon solution by the linear scattering for L_+ .

The uniqueness part requires some more work, since a priori we do not know if the solution is in the space X , globally in time. Let u be a solution on $[0, \infty)$ satisfying (3.11). Since it is bounded in the energy space, we can easily see that $N_\rho(t)$ is bounded. Therefore, it has to satisfy (3.4), and the reduced integral equation (3.5) as well as (3.6) for all $0 < t < \infty$. To see that $\lambda \in L_t^1(0, \infty)$ and $\gamma \in L_t^2 L_x^6(0, \infty)$, consider the norm

$$\|(\lambda, \gamma)\|_{X_T} := \|\lambda\|_{L^1(0,T)} + \|\lambda\|_{L^\infty(T,\infty)} + \|\gamma\|_{St(0,T)}, \quad (3.26)$$

for $T > 0$. The energy bound implies that $\|(\lambda, \gamma)\|_{X_T} < \infty$ for all $T > 0$, but we require a uniform bound. From the integral equations one concludes that

$$\begin{aligned} \|\lambda\|_{L^1(0,T) \cap L^\infty(T,\infty)} &\lesssim \|N_\rho\|_{L^1(0,T) \cap L^\infty(T,\infty)} \\ &\lesssim \nu + \nu(\|\lambda\|_{L^1(0,T) \cap L^\infty(T,\infty)} + \|\gamma\|_{St(0,T)}), \end{aligned} \quad (3.27)$$

and using the Strichartz estimate, one further has

$$\|\gamma\|_{St(0,T)} \lesssim \nu + \nu(\|\lambda\|_{L^1(0,T)} + \|\gamma\|_{St(0,T)}), \quad (3.28)$$

Thus we obtain by Fatou

$$\|(\lambda, \gamma)\|_X \leq \liminf_{T \rightarrow \infty} \|(\lambda, \gamma)\|_{X_T} \lesssim \nu, \quad (3.29)$$

and the contraction mapping principle implies the uniqueness. \square

As for Proposition 3.2, we only need to let \mathcal{M} be those $\vec{u}(0)$ for all the solutions u constructed above, satisfying (3.11) with $E_0 = \nu^2/C$.

4. ONE-PASS THEOREM

The key observation in our analysis of global dynamics is that any solution with energy only slightly higher than the ground states *can come close to the ground states at most once*. More precisely, if a solution u passes in and out of a small neighborhood of $\{\pm Q\}$, then it can never come back again. In particular, there is no homoclinic orbit connecting $\pm Q$ with themselves. Even though an orbit connecting Q and $-Q$ should be called heteroclinic, we are regarding it as homoclinic, by identifying $\pm Q$ as one point, since there will be no difference in precluding them by our argument. This point of view becomes more natural if one considers NLS, where the ground state is really a connected set (topologically a cylinder in the radial case) generated from Q by the invariant group action (the modulation and the scaling symmetries).

The key ideas to preclude homoclinic orbits are:

- When a solution comes very close to Q , but does not fall on the center-stable⁴ manifold, then it eventually leaves any small neighborhood of Q in such a way that the unstable, i.e., exponentially growing, mode dominates all the others (the stable and dispersive⁵ ones). This also applies to the negative time direction.
- For the solutions ejected from a small neighborhood of Q with a dominating velocity in the unstable direction, we can use the virial identity, after suitable localization, as a Lyapunov-type quantity.

⁴If one wishes not make any reference to Proposition 3.1, then the dichotomy expressed in this idea simply becomes the general “trapped by Q ” or “non-trapped by Q ” distinction, cf. Theorem 1.1.

⁵However, we do not control quantitatively what “eventually” means, nor do we need to. The dynamics that takes place before the exponential expansion dominates is very complicated and relies on an interplay between the different components, and the dispersive PDE behavior can be of the same order of magnitude as the other dynamics. We therefore treat the “pre-exit” dynamics as a black box.

The localized virial identity will be used in the following form. For a smooth function w cutting-off the outside of light cones, we have

$$V_w(t) := \langle wu_t | (x\nabla + \nabla x)u \rangle, \quad \dot{V}_w(t) = -K_2(u(t)) + \text{error}, \quad (4.1)$$

where the error term is due to the cut-off and bounded by the linear energy at t in the exterior region.

In order to use V_w as a Lyapunov functional, we need a lower bound on $|K_2(u)|$, or $|K_0(u)|$ if we use the convexity of L^2 norm instead. In Payne-Sattinger [32] and Kenig-Merle [25, 26], this is achieved solely by the variational structure, which is not sufficient by itself in our setting. Hence, our lower bound comes in two ways:

- While $\vec{u}(t)$ is away from but still close to $\pm Q$, we use the hyperbolic nature of the eigenmode λ (Lemma 4.2).
- While $\vec{u}(t)$ is not so close or really far away from $\pm Q$, we use the variational structure (Lemma 4.3).

Note that the bound by λ is not meaningful by itself once $\vec{u}(t)$ moves far away from $\pm Q$, since it is based on the linearization. The variational estimate is not useful close to $\pm Q$, even if we knew that $\vec{u}(t)$ does not really approach $\pm Q$, because that lower bound depends badly on the distance from $\pm Q$. However, for small $\varepsilon > 0$ these two estimates exhibit sufficient overlap of their regions of validity.

Now we introduce a nonlinear distance function to $\pm Q$, which seems best suited in order to exploit the hyperbolic dynamics together with the nonlinear energy structure. Let

$$u = \sigma[Q + v], \quad v = \lambda\rho + \gamma, \quad \gamma \perp \rho \quad (4.2)$$

for $\sigma = \pm$, and decompose the energy into the linearized part (3.7) and the higher order:

$$E(\vec{u}) - J(Q) + k^2\lambda^2 = \|\vec{v}\|_E^2 - C(v), \quad C(v) := \langle Q|v^3 \rangle + \|v\|_{L^4}^4/4. \quad (4.3)$$

There exists $0 < \delta_E \ll 1$ such that

$$\|\vec{v}\|_E \leq 4\delta_E \implies |C(v)| \leq \|\vec{v}\|_E^2/2. \quad (4.4)$$

Let χ be a smooth function on \mathbb{R} such that $\chi(r) = 1$ for $|r| \leq 1$ and $\chi(r) = 0$ for $|r| \geq 2$. We define

$$d_\sigma(\vec{u}) := \sqrt{\|\vec{v}\|_E^2 - \chi(\|\vec{v}\|_E/(2\delta_E))C(v)}. \quad (4.5)$$

It has the following properties

$$\|\vec{v}\|_E/2 \leq d_\sigma(\vec{u}) \leq 2\|\vec{v}\|_E, \quad d_\sigma(\vec{u}) = \|\vec{v}\|_E + O(\|\vec{v}\|_E^2), \quad (4.6)$$

$$d_\sigma(\vec{u}) \leq \delta_E \implies d_\sigma(\vec{u})^2 = E(\vec{u}) - J(Q) + k^2\lambda^2. \quad (4.7)$$

Henceforth, we shall always assume that \vec{u} is decomposed as in (4.2) such that

$$d_Q(\vec{u}) := \inf_{\pm} d_{\pm}(\vec{u}) = d_\sigma(\vec{u}), \quad (4.8)$$

where the choice of sign σ is unique as long as $d_Q(\vec{u}) \leq 2\delta_E$. We also set

$$\lambda_{\pm}(t) := \lambda(t) \pm \dot{\lambda}(t)/k, \quad (4.9)$$

the unstable/stable modes for $t \rightarrow \infty$ relative to the linearized hyperbolic evolution, see (3.4). First, we investigate the solutions which are close to $\pm Q$ but which are moving away.

4.1. Eigenmode dominance. The first observation is that the eigenmode λ becomes dominant in the energy and has a fixed sign, once \vec{u} is slightly away from the ground state, compared with the energy level. This is a static statement in the phase space \mathcal{H} , and an immediate consequence of the definition of d_Q .

Lemma 4.1. *For any $\vec{u} \in \mathcal{H}$ satisfying*

$$E(\vec{u}) < J(Q) + d_Q(\vec{u})^2/2, \quad d_Q(\vec{u}) \leq \delta_E, \quad (4.10)$$

one has $d_Q(\vec{u}) \simeq |\lambda|$.

In particular, λ has a fixed sign in each connected component of the above region.

Proof. (4.7) yields

$$d_Q(\vec{u})^2 = E(\vec{u}) - J(Q) + k^2\lambda^2 < d_Q(\vec{u})^2/2 + k^2\lambda^2. \quad (4.11)$$

and so, $k^2\lambda^2/16 \leq \|\vec{v}\|_E^2/8 \leq d_Q(\vec{u})^2/2 < k^2\lambda^2$. \square

4.2. Ejection process. The following lemma is the key to extract the hyperbolic nature from our PDE. Here the linearized evolution of the eigenmode λ plays the main role, and we specify that the solution is exiting rather than entering, by (4.13).

Lemma 4.2. *There exists a constant $0 < \delta_X \leq \delta_E$ with the following property. Let $u(t)$ be a local solution of NLKG on an interval $[0, T]$ satisfying*

$$R := d_Q(\vec{u}(0)) \leq \delta_X, \quad E(\vec{u}) < J(Q) + R^2/2 \quad (4.12)$$

and for some $t_0 \in (0, T)$,

$$d_Q(\vec{u}(t)) \geq R \quad (0 < \forall t < t_0). \quad (4.13)$$

Then $d_Q(\vec{u}(t))$ increases monotonically until reaching δ_X , and meanwhile,

$$\begin{aligned} d_Q(\vec{u}(t)) &\simeq -\mathfrak{s}\lambda(t) \simeq -\mathfrak{s}\lambda_+(t) \simeq e^{kt}R, \\ |\lambda_-(t)| + \|\vec{\gamma}(t)\|_E &\lesssim R + e^{2kt}R^2, \\ \min_{s=0,2} \mathfrak{s}K_s(u(t)) &\gtrsim d_Q(\vec{u}(t)) - C_*d_Q(\vec{u}(0)), \end{aligned} \quad (4.14)$$

for either $\mathfrak{s} = +1$ or $\mathfrak{s} = -1$, where $C_ \geq 1$ is a constant.*

Proof. Lemma 4.1 yields $d_Q(\vec{u}) \simeq |\lambda|$ as long as $R \leq d_Q(\vec{u}) \leq \delta_E$, whereas the energy conservation of NLKG and the equation of λ give as long as $d_Q(\vec{u}) \leq \delta_E$, see (3.7),

$$\partial_t d_Q(\vec{u})^2 = 2k^2\lambda\dot{\lambda}, \quad \partial_t^2 d_Q(\vec{u})^2 = 2k^2|\dot{\lambda}|^2 + 2k^4|\lambda|^2 + 2k^2\lambda N_\rho(v). \quad (4.15)$$

The exiting condition (4.13) implies $\partial_t d_Q(\vec{u})^2|_{t=0} \geq 0$. Since $N_\rho(v) \lesssim \|v\|_{H^1}^2$, we have $\partial_t^2 d_Q(\vec{u})^2 \simeq d_Q(\vec{u})^2$ as long as $d_Q(\vec{u}) \simeq |\lambda| \ll 1$.

Hence, imposing $\delta_X \leq \delta_E$ and small enough, we deduce that $d_Q(\vec{u}) \geq R$ strictly increases until it reaches δ_X ; meanwhile, $d_Q(\vec{u}) \simeq \mathfrak{s}\lambda$ for $\mathfrak{s} \in \{\pm 1\}$ fixed. Since

$$\lambda_+^2 - \lambda_-^2 = 4\lambda\dot{\lambda}/k \geq 0, \quad (4.16)$$

we also infer that $\lambda_+ \simeq \lambda$.

Next, integrating the equation (3.2) for λ yields

$$|\vec{\lambda}(t) - \vec{\lambda}_0(t)| \lesssim \int_0^t e^{k(t-s)} |N_\rho(v(s))| ds \lesssim \int_0^t e^{k(t-s)} |\lambda(s)|^2 ds, \quad (4.17)$$

where λ_0 denotes the linearized solution, which satisfies

$$|\lambda_0(t)| = |e^{kt}\lambda_+(0) + e^{-kt}\lambda_-(0)|/2 \simeq Re^{kt}. \quad (4.18)$$

Then by continuity in time we deduce

$$\lambda(t) \simeq -sRe^{kt}, \quad |\vec{\lambda}(t) - \vec{\lambda}_0(t)| \lesssim R^2 e^{2kt}, \quad (4.19)$$

as long as $Re^{kt} \ll 1$. This yields the upper bounds on λ_\pm as well.

To bound the remainder γ , we use the energy for the λ equation, see (4.3),

$$|\partial_t[-k^2\lambda^2/2 + \dot{\lambda}^2/2 - C(\lambda\rho)]| = |(N_\rho(v) - N_\rho(\lambda\rho))\dot{\lambda}| \lesssim \|\gamma\|_{H^1} |\lambda|^2. \quad (4.20)$$

Subtracting it from the energy (4.3) yields

$$|\partial_t[\|\vec{\gamma}\|_E^2 - C(v) + C(\lambda\rho)]| \lesssim \|\gamma\|_{H^1} |\lambda|^2. \quad (4.21)$$

Integrating this bound and using the bound on λ and $\vec{\gamma}(0)$, one obtains

$$\|\vec{\gamma}\|_{L_t^\infty E(0,T)}^2 \lesssim R^2 + \|\vec{\gamma}\|_{L_t^\infty E(0,T)} R^2 e^{2kT}, \quad (4.22)$$

which implies the desired bound on γ .

Finally, recall from (2.14) that

$$K_0(u) = -k^2\lambda\langle Q|\rho\rangle - \langle 2Q^3|\gamma\rangle + O(\|v\|_{H^1}^2), \quad (4.23)$$

and similarly we can expand K_2 around Q :

$$K_2(u) = -(k^2/2 + 2)\lambda\langle Q|\rho\rangle - \langle 2Q + Q^3|\gamma\rangle + O(\|v\|_{H^1}^2). \quad (4.24)$$

Since $\langle Q|\rho\rangle > 0$ by their positivity, we obtain the desired bound on K_s . \square

Note that the above proof did not use the equation for γ , see (3.2). Although we do have the full Strichartz estimate for the linearized evolution of γ at our disposal, thanks to the spectral gap condition (1.15), the above proof does not require any dispersive nature of γ , and so is applicable to more general cases. This is indeed natural since we are dealing with that part of dynamics dominated by the hyperbolicity in λ .

4.3. Variational lower bounds. Now we turn to the solutions away from $\pm Q$. The variational estimate is derived as an extension of Lemma 2.2. This is essentially a static statement, where $\vec{u} = (u, \dot{u})$ should be simply regarded as a point in the phase space \mathcal{H} .

Lemma 4.3. *For any $\delta > 0$, there exist $\varepsilon_0(\delta), \kappa_0, \kappa_1(\delta) > 0$ such that for any $\vec{u} \in \mathcal{H}$ satisfying*

$$E(\vec{u}) < J(Q) + \varepsilon_0(\delta)^2, \quad d_Q(\vec{u}) \geq \delta, \quad (4.25)$$

one has either

$$K_0(u) \leq -\kappa_1(\delta) \quad \text{and} \quad K_2(u) \leq -\kappa_1(\delta), \quad (4.26)$$

or

$$K_0(u) \geq \min(\kappa_1(\delta), \kappa_0 \|u\|_{H^1}^2) \quad \text{and} \quad K_2(u) \geq \min(\kappa_1(\delta), \kappa_0 \|\nabla u\|_{L^2}^2). \quad (4.27)$$

Proof. κ_0 is an absolute constant that will be determined via the constant in (2.8). First we prove the conclusion separately for $s = 0$ and $s = 2$ by contradiction. Fix $s = 0$ or $s = 2$ and $\delta > 0$, and suppose that there exists a sequence $\vec{u}_n \in \mathcal{H}$ satisfying (4.25) with $\varepsilon_0 = 1/n$ but neither (4.26) nor (4.27) with $\kappa_1 = 1/n$. Since $K_s(u_n)$ is bounded, the definition (2.5) implies that $G_s(u_n) \simeq \|u_n\|_{H^1}^2$ is also bounded, and so $K_s(u_n) \rightarrow 0$. Then by the same argument as in Lemma 2.1, we deduce that u_n converges, after extraction of a subsequence, strongly to 0 or $\pm Q$. In the latter case, (4.25) implies that

$$\delta^2 \leq \liminf_{n \rightarrow \infty} \|\dot{u}_n\|_{L^2}^2 \leq 2\varepsilon_0^2, \quad (4.28)$$

which is precluded by choosing $\varepsilon_0(\delta) < \delta/2$. If $u_n \rightarrow 0$, then (2.8) implies that the quadratic part dominates in $K_s(u_n)$ for large n , so (4.27) holds for some $\kappa_0 > 0$ independently of δ . Thus we obtain (4.26) or (4.27), separately for $s = 0$ and for $s = 2$.

It remains to show that they have the same sign. First note that

$$\begin{aligned} \tilde{\mathcal{K}}_s^+ &:= \{\vec{u} \in \mathcal{H} \mid E(\vec{u}) < J(Q) + \varepsilon_0^2, \ d_Q(\vec{u}) > \delta, \ K_s(u) \geq 0\}, \\ \tilde{\mathcal{K}}_s^- &:= \{\vec{u} \in \mathcal{H} \mid E(\vec{u}) < J(Q) + \varepsilon_0^2, \ d_Q(\vec{u}) > \delta, \ K_s(u) < 0\}, \end{aligned} \quad (4.29)$$

for $s = 0, 2$ are open sets satisfying

$$\tilde{\mathcal{K}}_s^+ \cap \tilde{\mathcal{K}}_s^- = \emptyset, \quad \tilde{\mathcal{K}}_0^+ \cup \tilde{\mathcal{K}}_0^- = \tilde{\mathcal{K}}_2^+ \cup \tilde{\mathcal{K}}_2^-. \quad (4.30)$$

Since $\tilde{\mathcal{K}}_0^+$ and $\tilde{\mathcal{K}}_2^+$ have the point 0 in common, it suffices to show that both are connected. For that purpose, we use two kinds of deformations in $\tilde{\mathcal{K}}_s^+$. Fix $s = 0, 2$ and take any $u \in \tilde{\mathcal{K}}_s^+$ satisfying $\delta < d_Q(\vec{u}) \leq 2\delta \ll \delta_E$. Recall the expansion

$$\begin{aligned} 2d_Q(\vec{u})^2 &= k^2\lambda^2 + \langle L_+\gamma | \gamma \rangle + \|\dot{v}\|_{L^2}^2 - 2C(v), \\ 2[E(\vec{u}) - J(Q)] &= -k^2\lambda^2 + \langle L_+\gamma | \gamma \rangle + \|\dot{v}\|_{L^2}^2 - 2C(v). \end{aligned} \quad (4.31)$$

Lemma 4.1 implies $|\lambda| \simeq d_Q(\vec{u})$ provided that we choose $\varepsilon_0(\delta)^2 < \delta^2/2$. We deform \vec{u} by increasing $|\lambda|$, while fixing γ and \dot{v} . Then $E(\vec{u})$ decreases and $d_Q(\vec{u})$ increases, as long as $d_Q(\vec{u}) \ll \delta_E$. Meanwhile, \vec{u} remains in $\tilde{\mathcal{K}}_s^+$ and eventually

$$E(\vec{u}) - J(Q) \lesssim -d_Q(\vec{u})^2 + O(\delta^2) + o(d_Q(u)^2) \ll -\delta^2. \quad (4.32)$$

Thus we can deform $\tilde{\mathcal{K}}_s^+ \cap \{d_Q(\vec{u}) \leq 2\delta\}$ into

$$\{\vec{u} \in \mathcal{H} \mid E(\vec{u}) - J(Q) \ll -\delta^2, K_s(u) \geq 0\} \subset \tilde{\mathcal{K}}_s^+, \quad (4.33)$$

which is contracted to $\{0\}$ by the scaling transform as in Lemma 2.2

$$\vec{u} = (u, \dot{u}) \mapsto (u_\nu^\nu, \nu \dot{u}) \quad (\nu : 1 \rightarrow +0). \quad (4.34)$$

For the remaining part of $\tilde{\mathcal{K}}_s^+$ we also use this scaling transform, until either reaching 0, or hitting the sphere $d_Q(\vec{u}) = 2\delta$, where it is reduced to the previous case. Thus we conclude that $\tilde{\mathcal{K}}_s^+$ for both s are connected and coincide. \square

4.4. Sign function away from the ground states. The above two lemmas enable us to define a sign functional away from $\pm Q$ by combining those of $-\lambda$ and K_s .

Lemma 4.4. *Let $\delta_S := \delta_X/(2C_*) > 0$ where δ_X and $C_* \geq 1$ are constants from Lemma 4.2. Let $0 < \delta \leq \delta_S$ and*

$$\mathcal{H}_{(\delta)} := \{\vec{u} \in \mathcal{H} \mid E(\vec{u}) < J(Q) + \min(d_Q(\vec{u})^2/2, \varepsilon_0(\delta)^2)\}, \quad (4.35)$$

where $\varepsilon_0(\delta)$ is given by Lemma 4.3. Then there exists a unique continuous function $\mathfrak{S} : \mathcal{H}_{(\delta)} \rightarrow \{\pm 1\}$ satisfying

$$\begin{cases} \vec{u} \in \mathcal{H}_{(\delta)}, d_Q(\vec{u}) \leq \delta_E & \implies \mathfrak{S}(\vec{u}) = -\text{sign}\lambda, \\ \vec{u} \in \mathcal{H}_{(\delta)}, d_Q(\vec{u}) \geq \delta & \implies \mathfrak{S}(\vec{u}) = \text{sign}K_0(u) = \text{sign}K_2(u), \end{cases} \quad (4.36)$$

where we set $\text{sign}0 = +1$ (a convention for the case $u = 0$).

Proof. Lemma 4.1 implies that $\text{sign}\lambda$ is continuous for $d_Q(\vec{u}) \leq \delta_E$, and Lemma 4.3 implies that $\text{sign}K_0(\vec{u}) = \text{sign}K_2(\vec{u})$ is continuous for $d_Q(\vec{u}) \geq \delta$. Hence, it suffices to see that they coincide at $d_Q(\vec{u}) = \delta_S \in [\delta, \delta_X]$ in $\mathcal{H}_{(\delta)}$. Let u be a solution of NLKG with $\vec{u}(0) \in \mathcal{H}_{(\delta)}$ and $d_Q(\vec{u}(0)) = \delta_S$. Then Lemma 4.2 implies that $\vec{u}(t)$ stays in $\mathcal{H}_{(\delta)}$ and $\text{sign}\lambda(t)$ is constant, until $d_Q(\vec{u}(t))$ reaches δ_X , which is after $-\text{sign}K_s(u(t))$ becomes the same as $\text{sign}\lambda(t)$, because $2C_*\delta_S \leq \delta_X$. Since $\text{sign}K_s(\vec{u})$ is constant for $d_Q(\vec{u}) \geq \delta$, we conclude that $\text{sign}\lambda(t) = -\text{sign}K_s(u(t))$ from the beginning $t = 0$. \square

The $\mathfrak{S} = +1$ side is uniformly bounded in the energy, as the following lemma shows.

Lemma 4.5. *There exists $M_* \sim J(Q)^{1/2}$ such that for any $\vec{u} \in \mathcal{H}_{(\delta_S)}$ satisfying $\mathfrak{S}(\vec{u}) = +1$ we have $\|\vec{u}\|_{\mathcal{H}} \leq M_*$.*

Proof. If $d_Q(\vec{u}) \leq \delta_S$, then

$$\|\vec{u}\|_{\mathcal{H}} \leq \|Q\|_{H^1} + \|\vec{v}\|_{\mathcal{H}} \lesssim J(Q)^{1/2} + d_Q(\vec{u})^{1/2} \lesssim J(Q)^{1/2}. \quad (4.37)$$

If $d_Q(\vec{u}) \geq \delta_S$, then $K_0(u) \geq 0$ together with (2.5) implies that

$$\|\vec{u}\|_{\mathcal{H}}^2 = 4E(\vec{u}) - K_0(u) \leq 4E(\vec{u}) \leq 4[J(Q) + \varepsilon_0(\delta_S)^2] \lesssim J(Q) \quad (4.38)$$

as desired. \square

4.5. Vanishing kinetic energy leads to scattering. Finally, we encounter the following problem in using the localized virial identity (4.1): in the regime $K_2(u(t)) \geq 0$ this functional can become arbitrarily small around 0 for the u component of \mathcal{H} . That is, $K_2(u(t))$ can vanish at some time if (and only if) $\|\nabla u(t)\|_{L_x^2}$ does so, see Lemma 4.3. Notice that we should treat this kind of vanishing only in the time averaged sense. The idea is then that all frequencies have to shift to 0, which leads to the scattering in both directions by the small Strichartz norm of subcritical regularity.

Lemma 4.6. *For any $M > 0$, there exists $\mu_0(M) > 0$ with the following property. Let $u(t)$ be a finite energy solution of NLKG (1.1) on $[0, 2]$ satisfying*

$$\|\vec{u}\|_{L_t^\infty(0,2;\mathcal{H})} \leq M, \quad \int_0^2 \|\nabla u(t)\|_{L^2}^2 dt \leq \mu^2 \quad (4.39)$$

for some $\mu \in (0, \mu_0]$. Then u extends to a global solution and scatters to 0 as $t \rightarrow \pm\infty$, and moreover $\|u(t)\|_{L_t^3 L_x^6(\mathbb{R} \times \mathbb{R}^3)} \ll \mu^{1/6}$.

Proof. First we see that u can be approximated by the free solution

$$v(t) := e^{i\langle \nabla \rangle t} v_+ + e^{-i\langle \nabla \rangle t} v_-, \quad v_{\pm} := [u(0) \mp i\langle \nabla \rangle^{-1} \dot{u}(0)]/2. \quad (4.40)$$

This follows simply from the Duhamel formula

$$\begin{aligned} \|v - u\|_{L_t^\infty H_x^1(0,2)} &\lesssim \|u^3\|_{L_t^1 L_x^2(0,2)} \lesssim \|u\|_{L_t^3 L_x^6}^3 \\ &\lesssim \|\nabla u\|_{L_t^2 L_x^2(0,2)}^2 \|\nabla u\|_{L_t^\infty L_x^2(0,2)} \leq \mu^2 M \ll \mu, \end{aligned} \quad (4.41)$$

if $\mu_0 M \ll 1$, where we used Hölder's inequality and the Sobolev embedding $\dot{H}^1 \subset L^6$. In particular,

$$\begin{aligned} 4\mu^2 &\geq \int_0^2 \|\nabla v(t)\|_{L_x^2}^2 dt \\ &= C \int |\xi|^2 \left[2|\widehat{v}_+|^2 + 2|\widehat{v}_-|^2 + \operatorname{Im}\{\langle \xi \rangle^{-1} (e^{4i\langle \xi \rangle} - 1) \widehat{v}_+ \widehat{v}_- \} \right] d\xi \\ &\gtrsim \|\nabla v_+\|_{L^2}^2 + \|\nabla v_-\|_{L^2}^2, \end{aligned} \quad (4.42)$$

where \widehat{v} denotes the Fourier transform in x of v . Now we use the Strichartz estimate for the free Klein-Gordon equation

$$\|e^{\pm i\langle \nabla \rangle t} \varphi\|_{L_t^3 B_{18/5,2}^{4/9}(\mathbb{R} \times \mathbb{R}^3)} \lesssim \|\varphi\|_{H_x^1}, \quad (4.43)$$

where $B_{p,q}^s$ denotes the Besov space with s regularity on L^p . Using the terminology of [22, Lemma 4.1], (4.43) means that $(\frac{1}{3}, \frac{5}{18}, \frac{4}{9})$ is 1-admissible. Combining it with Sobolev, we obtain

$$\begin{aligned} \|v\|_{L_t^3 L_x^6(\mathbb{R} \times \mathbb{R}^3)} &\lesssim \|v\|_{L_t^3 \dot{B}_{18/5,2}^{1/3}(\mathbb{R} \times \mathbb{R}^3)} \\ &\lesssim \sum_{\pm} \|v_{\pm}\|_{\dot{H}^{1/3} \cap \dot{H}^{8/9}} \lesssim M^{2/3} \mu^{1/3} + M^{1/9} \mu^{8/9} \ll \mu^{1/6}, \end{aligned} \quad (4.44)$$

if $\mu_0 M^4 \ll 1$. Therefore, we can identify u as the fixed point for the iteration in the global Strichartz norm

$$\|u\|_{L_t^\infty H_x^1(\mathbb{R} \times \mathbb{R}^3)} \lesssim M, \quad \|u\|_{L_t^3 L_x^6(\mathbb{R} \times \mathbb{R}^3)} \ll \mu^{1/6}, \quad (4.45)$$

which automatically scatters. \square

4.6. Local virial identity and non-existence of almost homoclinic orbits.

Using the constants in Lemmas 4.2–4.6, we choose $\varepsilon_*, \delta_*, R_*, \mu > 0$ such that

$$\begin{aligned} \delta_* &\leq \delta_S, \quad \delta_* \ll \delta_X, \quad \varepsilon_* \leq \varepsilon_0(\delta_*), \\ \varepsilon_* &\ll R_* \ll \min(\delta_*, \kappa_1(\delta_*)^{1/2}, \kappa_0^{1/2} \mu, J(Q)^{1/2}), \end{aligned} \quad (4.46)$$

$$\mu < \mu_0(M_*), \quad \mu^{1/6} \ll J(Q)^{1/2}. \quad (4.47)$$

Suppose that a solution $u(t)$ on the maximal existence interval $I \subset \mathbb{R}$ satisfies for some $\varepsilon \in (0, \varepsilon_*]$, $R \in (2\varepsilon, R_*]$, and $\tau_1 < \tau_2 < \tau_3 \in I$,

$$E(\vec{u}) < J(Q) + \varepsilon^2, \quad d_Q(\vec{u}(\tau_1)) < R < d_Q(\vec{u}(\tau_2)) > R > d_Q(\vec{u}(\tau_3)). \quad (4.48)$$

Then there exist $T_1 \in (\tau_1, \tau_2)$ and $T_2 \in (\tau_2, \tau_3)$ such that

$$d_Q(\vec{u}(T_1)) = R = d_Q(\vec{u}(T_2)) \leq d_Q(\vec{u}(t)) \quad (T_1 < t < T_2). \quad (4.49)$$

Lemma 4.4 gives us a fixed sign

$$\{\pm 1\} \ni \mathfrak{s} := \mathfrak{S}(u(t)) \quad (T_1 < t < T_2). \quad (4.50)$$

Now we derive the localized virial identity with a precise error bound. The cut-off function is defined by

$$w(t, x) = \begin{cases} \chi(x/(t - T_1 + S)) & (t < (T_1 + T_2)/2), \\ \chi(x/(T_2 - t + S)) & (t > (T_1 + T_2)/2), \end{cases} \quad (4.51)$$

where $S \gg 1$ is a constant to be determined later, and χ is a radial smooth function on \mathbb{R}^3 satisfying $\chi(x) = 1$ for $|x| \leq 1$ and $\chi(x) = 0$ for $|x| \geq 2$. Using the equation we have

$$V_w(t) := \langle wu_t | (x\nabla + \nabla x)u \rangle, \quad \dot{V}_w(t) = -K_2(u(t)) + O(E_{ext}(t)), \quad (4.52)$$

where $E_{ext}(t)$ denotes the exterior free energy defined by

$$E_{ext}(t) := \int_{X(t)} e^0(u) dx, \quad e^0(u) := [|\dot{u}|^2 + |\nabla u|^2 + |u|^2]/2, \quad (4.53)$$

$$x \in X(t) \iff \begin{cases} |x| > [t - T_1 + S] & (T_1 < t < \frac{T_1+T_2}{2}), \\ |x| > [T_2 - t + S] & (\frac{T_1+T_2}{2} < t < T_2). \end{cases}$$

We infer from the finite propagation speed that

$$\max_{j=1,2} E_{ext}(T_j) \ll 1 \implies \sup_{T_1 \leq t \leq T_2} E_{ext}(t) \lesssim \max_{j=1,2} E_{ext}(T_j), \quad (4.54)$$

because we can construct global solution v_j , such that $v_j = u$ in $X(t)$ for $|t - T_j| < |T_1 - T_2|/2$, and $\|\vec{v}(T_j)\|_{\mathcal{H}}^2 \lesssim E_{ext}(T_j)$, by cutting off the initial data at $t = T_j$ and using the small data theory. The exterior energy at $t = T_j$ is bounded by

$$E_{ext}(T_j) \lesssim e^{-2S} + \|\vec{\gamma}(T_j)\|_E^2, \quad (4.55)$$

where the term e^{-2S} is dominating the tails of Q and ρ , due to their exponential decay. Hence, choosing

$$S \gg |\log R| \gg 1, \quad (4.56)$$

we obtain $E_{ext}(t) \lesssim R^2$, and thus

$$\dot{V}_w(t) = -K_2(u(t)) + O(R^2) \quad (T_1 < t < T_2). \quad (4.57)$$

We turn to the leading term K_2 . In order to apply the ejection Lemma 4.2, we need the exiting property of the solution (4.13). For that purpose, take any $t_m \in [T_1, T_2]$ where $d_Q(\vec{u}(t))$ attains a minimum in t such that

$$(R \leq) R_m := d_Q(\vec{u}(t_m)) = \inf_{|t-t_m| < t_0, t \in [T_1, T_2]} d_Q(\vec{u}(t)) < \delta_*. \quad (4.58)$$

T_1 and T_2 obviously satisfy the above, but there may be numerous other minimum points if $\vec{u}(t)$ is circulating in the phase space, which indeed happens for the approximating ODE obtained by the projection onto the λ component.

Applying Lemma 4.2 to $u(t - t_m)$ and $u(t_m - t)$, as well as Lemma 4.4, one obtains

$$d_Q(\vec{u}(t)) \simeq -\mathfrak{s}\lambda(t) \simeq e^{k|t-t_m|} R_m, \quad \mathfrak{s}K_2(u(t)) \gtrsim d_Q(\vec{u}(t)) - C_* R_m, \quad (4.59)$$

until $d_Q(\vec{u}(t))$ reaches $\delta_X \gg \delta_*$. Let I_m denote that time interval around t_m . Then the integral of (4.57) is estimated on each I_m by

$$[\mathfrak{s}V_w(t)]_{I_m} \gtrsim \int_{I_m} [d_Q(\vec{u}(t)) - C_*R_m - O(R^2)]dt \sim \delta_X, \quad (4.60)$$

thanks to the exponential growth and $R \leq R_m \leq \delta_* \ll \delta_X$. In the remainder

$$I' := [T_1, T_2] \setminus \bigcup_m I_m, \quad (4.61)$$

one has $d_Q(\vec{u}(t)) > \delta_*$, and $\varepsilon \leq \varepsilon_0(\delta_*)$. Hence, Lemma 4.3 gives us (4.26) if $\mathfrak{s} = -1$, or (4.27) if $\mathfrak{s} = 1$.

In the latter case, Lemma 4.5 implies that $\|\vec{u}\|_{\mathcal{H}} \leq M_*$ on $[T_1, T_2]$. Since $d_Q(\vec{u}(t)) > \delta_* \gg R$ for any $t \in I'$, the hyperbolic behavior (4.59) on I_m implies

$$[t-1, t+1] \subset [T_1, T_2], \quad (4.62)$$

and Lemma 4.6 together with (4.47) implies

$$\int_{t-1}^{t+1} \|\nabla u(s)\|_{L_x^2}^2 ds > \mu^2, \quad (4.63)$$

since otherwise $\|u\|_{L_t^3 L_x^6} \ll \mu^{1/6} \ll J(Q)^{1/2}$, which contradicts that $d_Q(\vec{u}(T_1)) = R \ll J(Q)^{1/2}$. Combining it with (4.27) or (4.26), we obtain

$$\int_{t-1}^{t+1} K_2(u(s))ds \gtrsim \min(\kappa_1(\delta_*), \kappa_0\mu^2) \gg R_*^2, \quad (4.64)$$

due to the choice of R in (4.46). Combining this and (4.60), we obtain

$$[\mathfrak{s}V_w(t)]_{T_1}^{T_2} \gtrsim \delta_X \times \#\{t_m\}. \quad (4.65)$$

The left-hand side is bounded by using the exponential decay of Q ,

$$\lesssim \sum_{t=T_1, T_2} \|\dot{v}(t)\|_{L^2} + S\|v(t)\|_E^2 \lesssim R + SR^2 \lesssim R, \quad (4.66)$$

provided that we choose S such that

$$|\log R| \ll S \ll 1/R. \quad (4.67)$$

Thus we have arrived at a contradiction since $R \ll \delta_X$, precluding “almost homoclinic” orbits, i.e., any trajectory which exits from and returns to the R neighborhood of $\{\pm Q\}$. In other words, every solution is allowed to enter and exit a sufficiently small neighborhood of $\pm Q$ at most once.

Notice that the contradiction simply means that $T_2 = \infty$, and then all the above analysis remains valid, except for the upper bound of $[\mathfrak{s}V_w]_{T_1}^{T_2}$. Thus we have proven the following result.

Theorem 4.7 (One-pass theorem). *Let $\varepsilon_*, R_* > 0$ be as in (4.46). If a solution u of NLKG on an interval I satisfies for some $\varepsilon \in (0, \varepsilon_*]$, $R \in (2\varepsilon, R_*]$, and $\tau_1 < \tau_2 \in I$,*

$$E(\vec{u}) < J(Q) + \varepsilon^2, \quad d_Q(\vec{u}(\tau_1)) < R = d_Q(\vec{u}(\tau_2)), \quad (4.68)$$

then for all $t \in (\tau_2, \infty) \cap I =: I'$, we have $d_Q(\vec{u}(t)) > R$.

Moreover, there exist disjoint subintervals $I_1, I_2, \dots \subset I'$ with the following property: On each I_m , there exists $t_m \in I_m$ such that

$$d_Q(\vec{u}(t)) \simeq e^{k|t-t_m|} d_Q(\vec{u}(t_m)), \quad \min_{s=0,2} \mathfrak{s} K_s(u(t)) \gtrsim d_Q(\vec{u}(t)) - C_* d_Q(\vec{u}(t_m)), \quad (4.69)$$

where $\mathfrak{s} := \mathfrak{S}(\vec{u}(t)) \in \{\pm 1\}$ is constant, $d_Q(\vec{u}(t))$ is increasing for $t > t_m$, decreasing for $t < t_m$, equals to δ_X on ∂I_m . For each $t \in I' \setminus \bigcup_m I_m$ and $s = 0, 2$, one has $(t-1, t+1) \subset I'$, $d_Q(\vec{u}(t)) \geq \delta_*$ and

$$\int_{t-1}^{t+1} \min_{s=0,2} \mathfrak{s} K_s(u(t')) dt' \gg R_*^2. \quad (4.70)$$

By the monotonicity, we can keep applying the above theorem at each $t > \tau_2$ until $d_Q(\vec{u})$ reaches R_* . Besides, one concludes that at any later time $t_m > \tau_2$ necessarily $d_Q(\vec{u}) > R_*$. In other words, u cannot return to the distance R_* to $\pm Q$, after it is ejected to the distance $\delta_X > R_*$.

5. BLOWUP AFTER EJECTION

Here we prove that the solution u with $\mathfrak{S}(\vec{u}(\tau_2)) = -1$ in Theorem 4.7 blows up in finite time after τ_2 , by the contradiction argument of Payne-Sattinger, using K_0 . Suppose that u extends to all $t > \tau_2$ and let $y(t) := \|u(t)\|_{L^2}^2$. From the NLKG we have

$$\ddot{y} = 2[\|\dot{u}\|_{L_x^2}^2 + \mathfrak{s} K_0(u(t))]. \quad (5.1)$$

Applying the lower bound on K_0 in Theorem 4.7 to the integral yields

$$[y]_{\tau_2}^\infty \gtrsim \sum_{I_m} \delta_X + \int_{I'} R^2 dt = \infty, \quad (5.2)$$

and so $y(t) \rightarrow \infty$ as $t \rightarrow \infty$. Then from (5.1),

$$\ddot{y} \geq -8E(\vec{u}) + 6\|\dot{u}\|_{L^2}^2 + 2\|u\|_{H^1}^2 \geq 6\|\dot{u}\|_{L^2}^2 \geq 3(\dot{y})^2/(2y), \quad (5.3)$$

for large t , where we used Cauchy-Schwarz for $\dot{y} = 2\langle u|\dot{u} \rangle$. Hence,

$$\partial_t^2(y^{-1/2}) = -(2y^{3/2})^{-1}[y\ddot{y} - 3(\dot{y})^2/2] \leq 0, \quad (5.4)$$

which contradicts that $y \rightarrow \infty$ as $t \rightarrow \infty$. Therefore, u does not extend to $t \rightarrow \infty$.

6. SCATTERING AFTER EJECTION

For the solution u with $\mathfrak{S}(\vec{u}(\tau_2)) = +1$ in Theorem 4.7, the forward global existence follows from the energy bound Lemma 4.5. We prove its scattering to 0 for $t \rightarrow \infty$, by the contradiction argument of Kenig-Merle, using K_2 .

Fix $\varepsilon \in (0, \varepsilon_*)$ and let $\mathcal{U}(\varepsilon, R_*)$ be the collection of all solutions u of NLKG on $[0, \infty)$ satisfying

$$E(\vec{u}) \leq J(Q) + \varepsilon^2, \quad d_Q(\vec{u}[0, \infty)) \subset [R_*, \infty), \quad \mathfrak{S}(\vec{u}[0, \infty)) = +1. \quad (6.1)$$

Note that the first two conditions imply that $\vec{u}[0, \infty) \subset \mathcal{H}_{(\delta_*)}$ so that we can use Lemma 4.4 to define $\mathfrak{S}(\vec{u})$. By the remark after Theorem 4.7, any solution with $\mathfrak{S} = +1$ in that theorem will eventually satisfy the above conditions.

For each $E > 0$, let $M(E)$ be a uniform Strichartz bound defined by

$$M(E) := \sup\{\|u\|_{L_t^3 L_x^6(0,\infty)} \mid u \in \mathcal{U}(\varepsilon, R_*), E(\vec{u}) \leq E\}, \quad (6.2)$$

where we chose the norm $L_t^3 L_x^6$ to be an H^1 subcritical and non-sharp admissible Strichartz norm such that its finiteness implies scattering. We know by [22] that $M(E) < \infty$ for $E < J(Q)$. In fact, in that case a uniform bound holds for $L_t^3 L_x^6(\mathbb{R})$. In order to extend this property to $J(Q) + \varepsilon^2$, put

$$E^* = \sup\{E > 0 \mid M(E) < \infty\} \quad (6.3)$$

and assume towards a contradiction that

$$E^* < J(Q) + \varepsilon^2. \quad (6.4)$$

We consider the nonlinear profile decomposition for any sequence $u_n \in \mathcal{U}(\varepsilon, R_*)$ satisfying

$$E(u_n) \rightarrow E^*, \quad \|u_n\|_{L_t^3 L_x^6(0,\infty)} \rightarrow \infty. \quad (6.5)$$

We are going to show that the remainder in the decomposition is vanishing and there is only one profile which is a critical element, i.e.,

$$u_* \in \mathcal{U}(\varepsilon, R_*), \quad E(u_*) = E^*, \quad \|u_*\|_{L_t^3 L_x^6(0,\infty)} = \infty. \quad (6.6)$$

The decomposition is given as follows. First we have the *linear profile decomposition* of Bahouri-Gérard [1].

Proposition 6.1. *Let ψ_n be a sequence of free Klein-Gordon solutions bounded in \mathcal{H} . Then after replacing it by a subsequence, there exist a sequence of free solutions v^j bounded in \mathcal{H} , and sequences of times $t_n^j \in \mathbb{R}$ such that for v_n^j and γ_n^k defined by*

$$\psi_n(t) = \sum_{j < k} v_n^j(t) + \gamma_n^k(t), \quad v_n^j(t) = v^j(t + t_n^j), \quad (6.7)$$

we have for any $j < k$, $\tilde{\gamma}_n^k(-t_n^j) \rightarrow 0$ weakly in \mathcal{H} as $n \rightarrow \infty$,

$$\lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} \|\gamma_n^k\|_{(L_t^\infty L_x^4 \cap L_t^3 L_x^6)(\mathbb{R} \times \mathbb{R}^3)} = 0, \quad \lim_{n \rightarrow \infty} |t_n^j - t_n^k| = \infty. \quad (6.8)$$

The orthogonality of t_n^j and the weak vanishing of γ_n^k implies that

$$\limsup_{n \rightarrow \infty} \left| \|\vec{\psi}_n\|_{\mathcal{H}}^2 - \sum_{j < k} \|\vec{v}^j\|_{\mathcal{H}}^2 - \|\vec{\gamma}_n^k\|_{\mathcal{H}}^2 \right| = 0, \quad (6.9)$$

where the \mathcal{H} norms are independent of t because all components are free solutions, and in particular all of v^j and γ_n^k are uniformly bounded in \mathcal{H} . It is simpler than the original form of the Bahouri-Gérard decomposition for the wave equation, because the translational symmetry does not occur here by the radial assumption, and the frequency parameter is fixed by the subcriticality of the remainder estimate.

Proof. Since γ_n^k is bounded in H_x^1 , interpolation with the Strichartz bound implies that it suffices to estimate the remainder in $L_t^\infty L_x^4$. Let $\gamma_n^0 := \psi_n$ and $k = 0$. If

$$\nu^k := \limsup_{n \rightarrow \infty} \|\gamma_n^k\|_{L_t^\infty L_x^4} = 0, \quad (6.10)$$

then we are done by putting $\gamma_n^l = \gamma_n^k$ for all $l > k$. Otherwise, there exists a sequence $t_n^k \in \mathbb{R}$ such that $\|\gamma_n^k(-t_n^k)\|_{L_x^4} \geq \nu^k/2$ for large n . Since $\tilde{\gamma}_n^k(-t_n^k) \in \mathcal{H}$ is bounded,

after extracting a subsequence it converges weakly in \mathcal{H} , and $\gamma_n^k(-t_n^k)$ converges strongly in L_x^4 . Let v^k be the free solution given by the limit

$$\lim_{n \rightarrow \infty} \bar{\gamma}_n^k(-t_n^k) = \bar{v}^k(0), \quad (6.11)$$

then by Sobolev $\|v^k(0)\|_{H^1} \gtrsim \nu^k$. We repeat the same procedure by induction for $k = 1, 2, 3, \dots$. Let $U(t)$ denote the free Klein-Gordon propagator in \mathcal{H} . If $|t_n^j - t_n^k| \rightarrow c \in \mathbb{R}$ for some $j < k$, then

$$\bar{\gamma}_n^k(-t_n^k) = U(t_n^j - t_n^k) \bar{\gamma}_n^j(-t_n^j) \rightarrow 0, \quad (6.12)$$

weakly in \mathcal{H} , hence $|t_n^j - t_n^k| \rightarrow \infty$ as long as $v^k \neq 0$. Then for all $j \leq k$,

$$\bar{\gamma}_n^{k+1}(-t_n^j) = \bar{\gamma}_n^k(-t_n^j) - \bar{v}^k(t_n^k - t_n^j) \rightarrow 0 \quad (6.13)$$

weakly in \mathcal{H} . In particular we have (6.9), and so

$$\limsup_{n \rightarrow \infty} \|\psi_n\|_{\mathcal{H}}^2 \geq \sum_{j < k} \|v^j\|_{\mathcal{H}}^2 \gtrsim \sum_{j < k} (\nu^j)^2, \quad (6.14)$$

uniformly in k . Hence, $\limsup_{n \rightarrow \infty} \|\gamma_n^k\|_{L_t^\infty L_x^4} = \nu^k \rightarrow 0$, as $k \rightarrow \infty$. \square

Before applying the profile decomposition, we translate u_n in t to achieve

$$d_Q(\vec{u}_n(0)) > \frac{2}{3}\delta_X, \quad K_2(u_n(0)) \gg \varepsilon_*^2. \quad (6.15)$$

Since $d_Q(\vec{u}_n)$ remains above R_* , the ejection Lemma 4.2 implies that there exists $0 \leq T_n \lesssim k^{-1} \log(\delta_X/R_*)$ so that $d_Q(\vec{u}_n(T_n)) \geq \delta_X$. Since $\mathfrak{S} = +1$, Lemma 4.5 implies that $\|u_n\|_{L^\infty \mathcal{H}(0, \infty)} \leq M_*$. Since $\|u_n\|_{L_t^3 L_x^6} \rightarrow \infty$, by the same argument as for (4.64), we deduce that there exists $0 \leq T'_n$ near T_n such that

$$K_2(u_n(T'_n)) \gg R_*^2 > 2\varepsilon_*^2, \quad d_Q(\vec{u}_n(T'_n)) > \frac{2}{3}\delta_X. \quad (6.16)$$

Translating $u_n := u_n(t - T'_n)$, we obtain (6.15), in addition to (6.5).

Now apply the above lemma to the free solution with the same initial data as u_n , and let w_n^j be the nonlinear solution with the same data as v_n^j at $t = 0$

$$U(t)\vec{u}_n(0) = \sum_{j < k} \vec{v}_n^j + \bar{\gamma}_n^k, \quad \vec{w}_n^j(t) = U^N(t)\vec{v}_n^j(0), \quad (6.17)$$

where $U^N(t)$ denotes the nonlinear Klein-Gordon propagator in \mathcal{H} . Let $t_n^j \rightarrow t_\infty^j \in [-\infty, \infty]$ and w^j be the nonlinear solution satisfying

$$\lim_{t \rightarrow t_\infty^j} \|\vec{w}^j(t) - \vec{v}^j(t)\|_{\mathcal{H}} = 0, \quad (6.18)$$

which exists at least locally around $t = t_\infty^j$, as the unique solution of either the Cauchy problem at $t_\infty^j \in \mathbb{R}$ or the wave operator at $t_\infty^j \in \{\pm\infty\}$. As a consequence of the local theory one has

$$w_n^j(t) - w^j(t + t_n^j) \rightarrow 0 \quad (n \rightarrow \infty) \quad (6.19)$$

in the energy and Strichartz norms locally around $t = 0$. Thus we consider the following *nonlinear profile decomposition*

$$u_n = \sum_{j < k} u_n^j + \gamma_n^k + \text{error}, \quad u_n^j := w^j(t + t_n^j). \quad (6.20)$$

Then $K_2(u_n(0)) \gg \varepsilon^2$, $E(u_n) < J(Q) + \varepsilon^2$ and the orthogonality of the linear decomposition imply

$$\begin{aligned} J(Q) - \varepsilon^2 &> \limsup_{n \rightarrow \infty} [E(u_n) - K_2(u_n(0))/3] \\ &\geq \limsup_{n \rightarrow \infty} G_2(u_n(0)) = \limsup_{n \rightarrow \infty} \sum_{j < k} G_2(u_n^j(0)) + G_2(\gamma_n^j(0)). \end{aligned} \quad (6.21)$$

Since G_2 is positive definite, we obtain $\limsup_n G_2(u_n^j(0)) < J(Q)$, which implies, via the minimizing property (2.6), that $K_s(u_n^j(0)) \geq 0$ for large n and $s = 0, 2$. In particular, they all have positive energy, and so by the orthogonality applied to the energy, we deduce that $E(u^j) < J(Q)$ except for at most one of them. Those nonlinear profiles scatter as $t \rightarrow \pm\infty$.

If all of the profiles scatter for $t \rightarrow \pm\infty$, or more precisely if $\|u^j\|_{L_t^3 L_x^6(\mathbb{R})} < \infty$, then the following long-time perturbation argument implies that the original solutions u_n also scatter and remain bounded in the Strichartz norm uniformly in n .

Lemma 6.2. *There are continuous functions $\nu_0, C_0 : (0, \infty)^2 \rightarrow (0, \infty)$ such that the following holds: Let $I \subset \mathbb{R}$ be an interval, $u, w \in C(I; \mathcal{H})$ satisfying for some $A, B > 0$ and $t_0 \in I$*

$$\|\vec{u}\|_{L_t^\infty(I; \mathcal{H})} + \|\vec{w}\|_{L_t^\infty(I; \mathcal{H})} \leq A, \quad \|w\|_{L_t^3(I; L_x^6)} \leq B, \quad (6.22)$$

$$\|eq(u)\|_{L_t^1(I; L_x^2)} + \|eq(w)\|_{L_t^1(I; L_x^2)} + \|\gamma_0\|_{L_t^3(I; L_x^6)} \leq \nu_0(A, B), \quad (6.23)$$

where $eq(u) := \ddot{u} - \Delta u + u - u^3$, and $\vec{\gamma}_0 := U(t - t_0)(\vec{u} - \vec{w})(t_0)$. Then

$$\|\vec{u} - \vec{w} - \vec{\gamma}_0\|_{L_t^\infty(I; \mathcal{H})} + \|u - w\|_{L_t^3(I; L_x^6)} \leq C_0(A, B)\nu_0. \quad (6.24)$$

Proof. Let $Z := L_t^3 L_x^6$ and

$$\gamma := u - w, \quad e := (\partial_t^2 - \Delta + 1)(u - w) - u^3 + w^3. \quad (6.25)$$

There is a partition of the right half of I :

$$\begin{aligned} t_0 < t_1 < \dots < t_n, \quad I_j = (t_j, t_{j+1}), \quad I \cap (t_0, \infty) = (t_0, t_n), \\ \|w\|_{Z(I_j)} \leq \delta \quad (j = 0, \dots, n-1), \quad n \leq C(B, \delta). \end{aligned} \quad (6.26)$$

We omit the estimate on $I \cap (-\infty, t_0)$ since it is the same by symmetry. Let $\gamma_j(t) := U(t - t_j)\vec{\gamma}(t_j)$. Then the Strichartz estimate applied to the equations of γ and γ_{j+1} implies

$$\begin{aligned} \|\gamma - \gamma_j\|_{Z(I_j)} + \|\gamma_{j+1} - \gamma_j\|_{Z(\mathbb{R})} &\lesssim \|(w + \gamma)^3 - (w)^3 + e\|_{L_t^1 L_x^2(I_j)} \\ &\lesssim (\delta + \|\gamma\|_{Z(I_j)})^2 \|\gamma\|_{Z(I_j)} + \nu_0. \end{aligned} \quad (6.27)$$

Hence, by induction on j and continuity in t , one obtains

$$\|\gamma\|_{Z(I_j)} + \|\gamma_{j+1}\|_{Z(t_{j+1}, t_n)} \leq C[\|\gamma_j\|_{Z(t_j, t_n)} + \nu_0] \leq (2C)^j \nu_0 \leq (2C)^n \nu_0 \ll \delta, \quad (6.28)$$

provided that $\nu_0(A, B)$ is chosen small enough. Repeating the estimate (6.27) once more, we can bound the full Strichartz norms on γ . \square

The point in using the above perturbation lemma is that the orthogonality in (6.8), i.e., $|t_n^j - t_n^k| \rightarrow \infty$, implies that

$$\left(\sum_{j < k} u_n^j + \gamma_n^k\right)^3 = \sum_{j < k} (u_n^j)^3 + o(1), \quad (6.29)$$

in $L_t^1 L_x^2$ as $n \rightarrow \infty$ and $k \rightarrow \infty$, as soon as each u_n^j has globally finite $L_t^3 L_x^6$ norm. The same argument works even if the norm is finite only on $(0, \infty)$ or $(-\infty, 0)$, provided that the translation t_n^j and the interval I are placed such that the infinite part of the norm does not contribute. Hence at least one profile does not scatter. Let u^0 be the non-scattering profile.

If $t_\infty = +\infty$, then by the construction of the nonlinear profile, u^0 scatters as $t \rightarrow \infty$, leading to a uniform bound on $\|u_n\|_{L_t^3 L_x^6(0, \infty)}$ for large n , a contradiction.

If $t_\infty^0 = -\infty$, then u^0 scatters as $t \rightarrow -\infty$ by the same reason. Hence u^0 does not scatter as $t \rightarrow +\infty$. If $d_Q(\vec{u}^0(t)) > 3\varepsilon$ on its maximal existence interval, then $\mathfrak{S}(\vec{u}^0(t)) = +1$ is preserved from $t = -\infty$, so \vec{u}^0 is a global solution and a critical element, after time translation if necessary. Indeed, if it enters the interval $(3\varepsilon, R_*]$, but no deeper than that, then the ejection takes place and so the solution can never return to the R_* distance for later time. Suppose now that $d_Q(\vec{u}^0(t_*)) \leq 3\varepsilon$ for some $t_* \in \mathbb{R}$. Then $\|u^0\|_{L_t^3 L_x^6(-\infty, t_*)} < \infty$ implies that (6.29) holds in $L_t^1 L_x^2(-\infty, t_* - t_n^0)$. Hence, the nonlinear profile decomposition becomes a good approximation of u_n on that interval. Then taking k and n large enough, we infer from the perturbation lemma

$$d_Q(\vec{u}_n(t_* - t_n^0)) \leq d_Q(\vec{u}^0(t_*)) + O(\varepsilon) \lesssim \varepsilon \ll R_*, \quad (6.30)$$

contradicting that $d_Q(\vec{u}_n(0, \infty)) \subset [R_*, \infty)$, since $t_* - t_n^0 \rightarrow \infty$.

The remaining case $t_\infty^0 \in \mathbb{R}$ is also similar. If u^0 scatters for $t \rightarrow \infty$, then the perturbation lemma implies that $\|u_n\|_{L_t^3 L_x^6(0, \infty)}$ is bounded for large n . Hence, u^0 does not scatter for $t \rightarrow \infty$. On the other hand, the decomposition at $t = 0$ gives

$$\frac{2}{3}\delta_X \leq \limsup_{n \rightarrow \infty} d_Q(\vec{u}_n(0)) \leq d_Q(\vec{u}^0(t_\infty^0)) + O(\varepsilon), \quad (6.31)$$

and so $d_Q(\vec{u}^0(t_\infty^0)) > \delta_X/2 > \delta_S \gg R_*$. Moreover,

$$K_s(\vec{u}^0(t_\infty^0)) = \lim_{n \rightarrow \infty} K_s(\vec{u}^0(t_n^0)) \geq 0, \quad (6.32)$$

and so we deduce from Lemma 4.4 that $\mathfrak{S}(\vec{u}^0(t_\infty^0)) = +1$.

By the same argument as above, u^0 is either a critical element or $d_Q(\vec{u}^0(t_*)) \leq 3\varepsilon$ for some $t_* > t_\infty^0$. Then (6.31) together with the above theorem implies that $d_Q(\vec{u}^0(t)) \geq R_*$ for $t \leq t_\infty^0$. If u^0 does not scatter as $t \rightarrow -\infty$, then $u^0(t_\infty^0 - t)$ is a critical element. Suppose that u^0 scatters as $t \rightarrow -\infty$. Then the profile decomposition is a good approximation of u_n on $(-\infty, t_* - t_n^0)$ for large n . Then, by the perturbation lemma for large k and n ,

$$\varepsilon \gtrsim d_Q(\vec{u}^0(t_*)) + O(\varepsilon) \geq \limsup_{n \rightarrow \infty} d_Q(\vec{u}_n(t_* - t_\infty^0)) \geq R_*, \quad (6.33)$$

which contradicts $\varepsilon \ll R_*$.

Thus in conclusion, for some $T \in \mathbb{R}$ and $\mathfrak{s} \in \{\pm 1\}$, $u^0(\mathfrak{s}t + T)$ is a critical element. Since u_n is a minimizing sequence, it implies also that the other components must

vanish strongly in the linear profile decomposition. Therefore, on a subsequence,

$$\lim_{n \rightarrow \infty} \|\vec{u}_n(T'_n) - \vec{u}^0(t_n^0)\|_{\mathcal{H}} = 0, \quad (6.34)$$

where $T'_n \geq 0$ is the time shift for (6.15). Both T'_n and t_n^0 are bounded from above as $n \rightarrow \infty$. If $t_n^0 \rightarrow -\infty$ then u^0 scatters for $t \rightarrow -\infty$, and so $\|u_n\|_{L_t^3 L_x^6(-\infty, T'_n)}$ is bounded for large n , by the local theory of the wave operator.

Applying this to the sequence of solutions $u_n := u^0(t + \tau_n)$ for arbitrary $\tau_n \rightarrow \infty$, one obtains the precompactness of the forward trajectory of the critical element, and a contradiction from a localized (time-independent) virial identity together with the lower bound on K_2 . These steps are simpler than those in Kenig-Merle [26] (see [22] for NLKG). Thus we conclude that no solution u satisfies (6.6), and therefore $E^* = J(Q) + \varepsilon^2$.

Theorem 6.3. *For each $\varepsilon \in (0, \varepsilon_*]$, there exists $0 < M(J(Q) + \varepsilon^2) < \infty$ such that if a solution u of NLKG on $[0, \infty)$ satisfies $E(\vec{u}) \leq J(Q) + \varepsilon^2$, $d_Q(\vec{u}(t)) \geq R_*$ and $\mathfrak{S}(\vec{u}(t)) = +1$ for all $t \geq 0$, then u scatters to 0 as $t \rightarrow \infty$ and $\|u\|_{L_t^3 L_x^6(0, \infty)} \leq M$.*

Note that the uniform Strichartz bound is valid only on the time interval where the solution is already away from $\pm Q$ by a fixed distance. Without it, one cannot have any uniform bound even for those solutions scattering for both $t \rightarrow \pm\infty$, since they can stay close to $\pm Q$ for arbitrarily long time.

7. CLASSIFICATION OF THE GLOBAL BEHAVIOR, PROOF OF THEOREM 1.1

Fix $0 < \varepsilon \leq \varepsilon_*$ and let $\mathcal{H}^\varepsilon := \{\vec{u} \in \mathcal{H} \mid E(\vec{u}) < J(Q) + \varepsilon^2\}$ be the initial data set. We can define the following subsets according to the global behavior of the solution $u(t)$ to the NLKG equation: for $\sigma = \pm$ respectively,

$$\begin{aligned} \mathcal{S}_\sigma^\varepsilon &= \{\vec{u}(0) \in \mathcal{H}^\varepsilon \mid u(t) \text{ scatters as } \sigma t \rightarrow \infty\}, \\ \mathcal{T}_\sigma^\varepsilon &= \{\vec{u}(0) \in \mathcal{H}^\varepsilon \mid u(t) \text{ trapped by } \{\pm Q\} \text{ for } \sigma t \rightarrow \infty\}, \\ \mathcal{B}_\sigma^\varepsilon &= \{\vec{u}(0) \in \mathcal{H}^\varepsilon \mid u(t) \text{ blows up in } \sigma t > 0\}. \end{aligned} \quad (7.1)$$

The trapping for $\mathcal{T}_+^\varepsilon$ can be characterized as follows, with any $R \in (2\varepsilon, R_*)$:

$$\exists T > 0, \quad \forall t > T, \quad d_Q(\vec{u}(t)) < R. \quad (7.2)$$

Obviously those sets are increasing in ε , and have the conjugation property

$$X_\mp^\varepsilon = \{(u(0), -\dot{u}(0)) \in \mathcal{H}^\varepsilon \mid \vec{u}(0) \in X_\pm^\varepsilon\}, \quad (7.3)$$

for $X = \mathcal{S}, \mathcal{T}, \mathcal{B}$. Moreover, \mathcal{S}_+ and \mathcal{T}_+ are forward invariant by the flow of NLKG, while \mathcal{S}_- and \mathcal{T}_- are backward invariant. We have proven in the previous sections that

$$\mathcal{H}^\varepsilon = \mathcal{S}_+^\varepsilon \cup \mathcal{T}_+^\varepsilon \cup \mathcal{B}_+^\varepsilon = \mathcal{S}_-^\varepsilon \cup \mathcal{T}_-^\varepsilon \cup \mathcal{B}_-^\varepsilon, \quad (7.4)$$

the disjoint union for each sign. It follows from the scattering theory that $\mathcal{S}_\pm^\varepsilon$ are open. We claim the same for $\mathcal{B}_\pm^\varepsilon$, which is not a general fact. By the energy estimate

$$\|\vec{u}(t)\|_{L_t^\infty \mathcal{H}(0, T)} \lesssim \|\vec{u}(0)\|_{\mathcal{H}} + T \|u\|_{L_t^\infty H_x^1(0, T)}^3, \quad (7.5)$$

we deduce that if $t = T^*$ is the blow-up time then

$$\|u(t)\|_{\mathcal{H}} \gtrsim |T^* - t|^{-1/2}. \quad (7.6)$$

Hence, from the integral inequality

$$\partial_t^2 \|u\|_{L_x^2}^2 = 2[\|\dot{u}\|_{L_x^2}^2 - K_0(u(t))] \geq 6\|\dot{u}\|_{L_x^2}^2 + 2\|u\|_{H_x^1}^2 - 8E(\vec{u}), \quad (7.7)$$

we deduce that $\partial_t \|u\|_{L_x^2}^2 = 2\langle u|\dot{u} \rangle \rightarrow \infty$ as well as $K_0(u(t)) \rightarrow -\infty$ as $t \rightarrow T^* - 0$. We claim that if $T^{**} < T^*$ is very close to blow-up time, then every solution starting in $B_1(u(T^{**}))$, the unit-ball around $u(T^{**})$ relative to \mathcal{H} , necessarily blows up in the positive time direction. By the Payne-Sattinger argument, we only need to show that for any such solution, $K_0(u(t)) \leq -\kappa < 0$ for as long as it is defined. It is clear that this condition will hold initially. By Lemma 4.3 with the choice of ε_* in (4.46), we further see that it can be violated only if the solution returns to a neighborhood of $\pm Q$ of size δ_S . However, in that case necessarily $|\langle u|\dot{u} \rangle| \lesssim 1$, which is impossible since $\langle u|\dot{u} \rangle$ starts off very large and has to increase as long as $K_0(u(t)) < 0$. Thus, any solution close to u around $t = T^{**}$ cannot come close to $\pm Q$ and has to blow up sooner or later, as claimed. Therefore, $\mathcal{B}_\pm^\varepsilon$ are also open, so $\mathcal{T}_\pm^\varepsilon$ are relatively closed in \mathcal{H}^ε .

Since $\mathcal{B}_+^\varepsilon$ and $\mathcal{S}_+^\varepsilon$ are disjoint open, they are separated by $\mathcal{T}_+^\varepsilon$, that is, any two points from $\mathcal{S}_+^\varepsilon$ and $\mathcal{B}_+^\varepsilon$ cannot be joined by a curve without passing through $\mathcal{T}_+^\varepsilon$.

In a small ball around $\pm Q$, it is easy to see by means of the linearized flow that the open intersections $\mathcal{B}_\pm^\varepsilon \cap \mathcal{S}_\mp^\varepsilon$, $\mathcal{B}_+^\varepsilon \cap \mathcal{B}_-^\varepsilon$ and $\mathcal{S}_+^\varepsilon \cap \mathcal{S}_-^\varepsilon$ are all non-empty for any $\varepsilon > 0$. To obtain a solution u in $\mathcal{B}_+^\varepsilon \cap \mathcal{S}_-^\varepsilon$, choose initial data such that

$$\lambda(0) = 0, \quad \dot{\lambda}(0) = k\theta\varepsilon, \quad \vec{\gamma}(0) = 0, \quad (7.8)$$

for some $0 < \theta \ll 1$. Then the energy constraint $E(\vec{u}) < J(Q) + \varepsilon^2$ is satisfied, and by the same argument as in Lemma 4.2, we have

$$|\lambda(t) - \lambda_0(t)| \lesssim e^{2k|t|}\theta^2\varepsilon^2, \quad \|\gamma(t)\|_E \lesssim \theta\varepsilon + e^{2k|t|}\theta^2\varepsilon^2, \quad (7.9)$$

as long as $e^{k|t|}\theta\varepsilon \ll \delta_X$, where the free solution λ_0 is given in this case

$$\lambda_0(t) = \sinh(kt)\theta\varepsilon. \quad (7.10)$$

Hence, $u(t)$ exits the R_* neighborhood of $\pm Q$ both in $\pm t > 0$. Since $\lambda(t)t > 0$ for $\theta\varepsilon \ll e^{k|t|}\theta\varepsilon \ll \delta_X$, we obtain $\mathfrak{S}(\vec{u}(t))t < 0$ after exiting, and so u blows up in $t > 0$ and scatters for $t \rightarrow -\infty$. Therefore, $\mathcal{B}_\pm^\varepsilon \cap \mathcal{S}_\mp^\varepsilon$ are both non-empty.

In the same way, we construct solutions in $\mathcal{B}_+^\varepsilon \cap \mathcal{B}_-^\varepsilon$ and $\mathcal{S}_+^\varepsilon \cap \mathcal{S}_-^\varepsilon$, respectively. Let $u^{(\pm)}$ be those solutions for which λ are approximated by the free solutions

$$\lambda_0^{(\pm)}(t) = \pm \cosh(kt)\theta\varepsilon. \quad (7.11)$$

Then $u^{(+)} \in \mathcal{B}_+^\varepsilon \cap \mathcal{B}_-^\varepsilon$ and $u^{(-)} \in \mathcal{S}_+^\varepsilon \cap \mathcal{S}_-^\varepsilon$.

For $\varepsilon \ll e^{kt}\theta\varepsilon \ll \varepsilon^{1/2}$, the distances from the above solution $u \in \mathcal{B}_+^\varepsilon \cap \mathcal{S}_-^\varepsilon$ and from $\{\pm Q\}$ are estimated by

$$\|\vec{u}(t) - \vec{u}^{(+)}(t)\|_E \lesssim e^{2kt}\theta^2\varepsilon^2 + \theta\varepsilon \ll \varepsilon, \quad d_Q(\vec{u}(t)) \gtrsim e^{kt}\theta\varepsilon \gg \varepsilon \quad (7.12)$$

Hence, there exists a curve $\Gamma \subset \mathcal{H}^\varepsilon$ joining $u(t)$ and $u^{(+)}(t)$ within the region $\mathfrak{S}(\vec{u}) < 0$ and $d_Q(\vec{u}) \gg \varepsilon$. Since $u(t) \in \mathcal{S}_-^\varepsilon$ and $u^{(+)}(t) \in \mathcal{B}_-^\varepsilon$, there exists a point $p_0 \in \mathcal{T}_-^\varepsilon \cap \Gamma$. Since the solution starting from p_0 enters the 3ε -ball around $\pm Q$ as $t \rightarrow -\infty$, and initially p_0 is much further away and $\mathfrak{S}(p_0) < 0$, we conclude by the one-pass theorem that $p \in \mathcal{B}_+^\varepsilon$. Hence, $\mathcal{T}_-^\varepsilon \cap \mathcal{B}_+^\varepsilon$ is non-empty as well. In the same way, we can find a

point on the curve connecting $u(t)$ and $u^{(-)}(t)$ for some $t < 0$, which is in $\mathcal{T}_+^\varepsilon \cap \mathcal{S}_-^\varepsilon$. Therefore, $\mathcal{T}_\pm^\varepsilon \cap \mathcal{B}_\mp^\varepsilon$ and $\mathcal{T}_\pm^\varepsilon \cap \mathcal{S}_\mp^\varepsilon$ are both not empty. Taking the limit $\varepsilon \rightarrow +0$, it is easy to observe that they contain infinitely many points on different energy levels.

Finally, $\mathcal{T}_+^\varepsilon \cap \mathcal{T}_-^\varepsilon \ni Q$ is of course not empty, but there are infinitely many points besides Q , which can be seen by restricting \mathcal{H}^ε to the hyperplane $\dot{u} = 0$:

$$\mathcal{H}_{\dot{u}=0}^\varepsilon := \{(u, 0) \mid J(u) < J(Q) + \varepsilon^2\} \ni u^{(\pm)}(0). \quad (7.13)$$

Since $u^{(+)}(0) \in \mathcal{B}_\varepsilon^+$ and $u^{(-)}(0) \in \mathcal{S}_\varepsilon^+$, and there are infinitely many connecting curves in $\mathcal{H}_{\dot{u}=0}^\varepsilon$, there are infinitely many points in $\mathcal{T}_+^\varepsilon \cap \mathcal{H}_{\dot{u}=0}^\varepsilon$ separating $\mathcal{B}_+^\varepsilon$ and $\mathcal{S}_+^\varepsilon$. The symmetry $u(t) \mapsto u(-t)$ implies that

$$\mathcal{T}_+^\varepsilon \cap \mathcal{H}_{\dot{u}=0}^\varepsilon \subset \mathcal{T}_+^\varepsilon \cap \mathcal{T}_-^\varepsilon. \quad (7.14)$$

Thus we have shown that the nine solution sets are all non-empty.

By using the scattering theory, we can prove that $\mathcal{S}_+^\varepsilon$ and the energy section

$$\mathcal{S}_+^{\varepsilon-} := \{\vec{u} \in \mathcal{H} \mid U^N(t)\vec{u} \text{ scatters as } t \rightarrow \infty, E(\vec{u}) = J(Q) + \varepsilon^2\} \quad (7.15)$$

are both pathwise connected. To see this, let $\vec{u}_j \in \mathcal{S}_+^\varepsilon$ or $\mathcal{S}_+^{\varepsilon-}$ for $j = 0, 1$ and let \vec{u}_{j+} be their asymptotic profiles

$$\|U^N(t)\vec{u}_j - U(t)\vec{u}_{j+}\|_E \rightarrow 0 \quad (t \rightarrow \infty). \quad (7.16)$$

There exists a continuous curve $\vec{u}_{\theta+} : [0, 1] \rightarrow \mathcal{H}$ such that

$$\|\vec{u}_{\theta+}\|_{\mathcal{H}}^2 = (1 - \theta)\|\vec{u}_{0+}\|_{\mathcal{H}}^2 + \theta\|\vec{u}_{1+}\|_{\mathcal{H}}^2, \quad (7.17)$$

as well as u_θ solving NLKG on some (T_θ, ∞) such that

$$\|\vec{u}_\theta(t) - U(t)\vec{u}_{\theta+}\|_E \rightarrow 0 \quad (t \rightarrow \infty). \quad (7.18)$$

Moreover, T_θ and $\vec{u}_\theta(T_\theta + 1)$ are also continuous in θ . The scattering property implies that $\vec{u}_\theta(t) \in \mathcal{S}_+$ for any $t > T_\theta$, and

$$E(\vec{u}_\theta) = \|\vec{u}_{\theta+}\|_{\mathcal{H}}^2/2 = (1 - \theta)E(\vec{u}_0) + \theta E(\vec{u}_1). \quad (7.19)$$

Let $T := \sup_{0 \leq \theta \leq 1} T_\theta + 1 < \infty$, then \vec{u}_j and $\vec{u}_j(T)$ are connected by the flow, while $\vec{u}_0(T)$ and $\vec{u}_1(T)$ are connected by $\vec{u}_\theta(T)$, all included in $\mathcal{S}_+^\varepsilon$ or $\mathcal{S}_+^{\varepsilon-}$. This proves the connectedness of those sets.

So far, all arguments of this section have been independent of the spectral gap property (2.22). Now let us assume it for further investigation. Then $\mathcal{T}_+^\varepsilon$ contains the center-stable manifold for the $t \geq 0$ direction, as constructed in Proposition 3.1. In fact, it consists of the maximal backward extension of all solutions on this manifold. The linear approximation is the hyperplane $k\lambda = -\dot{\lambda}$. Hence, the classification into those nine sets looks like \otimes in the $(\lambda, \dot{\lambda})$ phase plain around Q . Since the flow is C^∞ in both directions, $\mathcal{T}_\pm^\varepsilon$ are indeed 1-codimensional and smooth manifolds in the energy space. Moreover, a small ball around each point of $\mathcal{T}_+^\varepsilon$ is separated by the hypersurface $\mathcal{T}_+^\varepsilon$ into $\mathcal{S}_+^\varepsilon$ and $\mathcal{B}_+^\varepsilon$.

Next we prove that $\mathcal{T}_+^\varepsilon$ consists of two connected components $\mathcal{T}_{+(\pm)}^\varepsilon$ defined by

$$\mathcal{T}_{+(\sigma)}^\varepsilon = \{\vec{u} \in \mathcal{H}^\varepsilon \mid U^N(t)\vec{u} \text{ is trapped by } \sigma Q \text{ as } t \rightarrow \infty\}. \quad (7.20)$$

To see that $\mathcal{T}_{+(+)}^\varepsilon$ is pathwise connected, take any couple of points and send them by the flow to the future where they are in 3ε distance from Q . Let \vec{u}_j with $j = 0, 1$ be those two after translation, and let $\vec{\gamma}_j$ be their asymptotic profiles

$$\|U^N(t)\vec{u}_j - \vec{Q} - U(t)\vec{\gamma}_j\|_E \rightarrow 0 \quad (t \rightarrow \infty), \quad E(\vec{u}_j) = J(Q) + \|\gamma_j\|_{\mathcal{H}}^2/2. \quad (7.21)$$

There is a continuous $\vec{\gamma}_\theta : [0, 1] \rightarrow \mathcal{H}$ such that $\|\vec{\gamma}_\theta\|_{\mathcal{H}}^2 = (1 - \theta)\|\vec{\gamma}_0\|_{\mathcal{H}}^2 + \theta\|\vec{\gamma}_1\|_{\mathcal{H}}^2$. Let $\lambda_\theta := (1 - \theta)\langle u_0 - Q | \rho \rangle + \theta\langle u_1 - Q | \rho \rangle$. Proposition 3.1 gives a continuous family of NLKG solutions u_θ such that $\langle u_\theta(0) - Q | \rho \rangle = \lambda_\theta$ and $\vec{u}_\theta(t) - \vec{Q} - U(t)\vec{\gamma}_\theta \rightarrow 0$ as $t \rightarrow \infty$. Then \vec{u}_0 and \vec{u}_1 are connected by the curve $\vec{u}_\theta(0)$ in $\mathcal{T}_{+(+)}^\varepsilon$.

To see the separation of $\mathcal{T}_{+(+)}^\varepsilon$ from $\mathcal{T}_{+(-)}^\varepsilon$, suppose for contradiction that p is a common point of their boundaries. Then $p \in \mathcal{T}_+^{2\varepsilon}$ by its relative closedness, so $U^N(t)p$ scatters to σQ with $\sigma = \pm$. Then its small neighborhood can be mapped by the flow into a ball around σQ . But it contains a point in $\mathcal{T}_{+(-\sigma)}^\varepsilon$, contradicting the one-pass theorem. Thus $\mathcal{T}_+^\varepsilon$ consists of two connected components $\mathcal{T}_{+(\pm)}^\varepsilon$.

We can observe also that

$$\partial\mathcal{S}_+^\varepsilon = \mathcal{T}_+^\varepsilon \cup \mathcal{T}_+^{=\varepsilon} \cup \mathcal{S}_+^{=\varepsilon} \quad (7.22)$$

is connected. \subset follows from that $\mathcal{B}_+^\varepsilon$ and $\{E(u) > J(Q) + \varepsilon^2\}$ are open. \supset is easily seen by perturbing the data from the right. The connectedness follows from that of the following three sets

$$\mathcal{T}_{+(+)}^{<} \cup \mathcal{T}_{+(+)}^{=\varepsilon}, \quad \mathcal{S}_+^{=\varepsilon} \cup \mathcal{T}_+^{=\varepsilon}, \quad \mathcal{T}_{+(-)}^{<} \cup \mathcal{T}_{+(-)}^{=\varepsilon}. \quad (7.23)$$

Kenig-Merle's solutions [25, 26] can be classified in our notation, see (7.1), by

$$\mathcal{S}_+^0 = \mathcal{S}_-^0, \quad \mathcal{B}_+^0 = \mathcal{B}_-^0, \quad \mathcal{T}_+^0 = \mathcal{T}_-^0 = \emptyset \quad (7.24)$$

(see [22] for the case of NLKG). This statement essentially follows from our theorems, because the energy constraint $E(\vec{u}) < E(Q)$ prohibits the sign change of K_s and scattering to $\pm Q$. The threshold case Theorem 1.4, i.e., the analogue of Duyckaerts-Merle [13, 14], requires a little more investigation in order to establish the uniqueness of the solutions in (5)-(9).

Proof of Theorem 1.4. Let $\widetilde{X}_\pm^0 = \bigcap_{\varepsilon>0} X_\pm^\varepsilon$ for $X = \mathcal{S}, \mathcal{B}, \mathcal{T}$. Then Theorem 1.4 can be restated as

$$\begin{aligned} \widetilde{\mathcal{S}}_+^0 \cap \widetilde{\mathcal{B}}_-^0 &= \widetilde{\mathcal{S}}_-^0 \cap \widetilde{\mathcal{B}}_+^0 = \emptyset, \quad \widetilde{\mathcal{T}}_+^0 \cap \widetilde{\mathcal{T}}_-^0 = \{\pm Q(t-s) \mid s \in \mathbb{R}\}, \\ \widetilde{\mathcal{S}}_+^0 \cap \widetilde{\mathcal{T}}_-^0 &= \{\pm W_-(t-s) \mid s \in \mathbb{R}\}, \quad \widetilde{\mathcal{B}}_+^0 \cap \widetilde{\mathcal{T}}_-^0 = \{\pm W_+(t-s) \mid s \in \mathbb{R}\}, \\ \widetilde{\mathcal{S}}_-^0 \cap \widetilde{\mathcal{T}}_+^0 &= \{\pm W_-(s-t) \mid s \in \mathbb{R}\}, \quad \widetilde{\mathcal{B}}_-^0 \cap \widetilde{\mathcal{T}}_+^0 = \{\pm W_+(s-t) \mid s \in \mathbb{R}\}, \end{aligned} \quad (7.25)$$

for some solutions W_\pm . To see that $\widetilde{\mathcal{S}}_+^0 \cap \widetilde{\mathcal{B}}_-^0 = \emptyset$, let u be a solution in $\widetilde{\mathcal{S}}_+^0 \cap \widetilde{\mathcal{B}}_-^0$. Theorem 1.2 implies that there exists a finite interval $I_\varepsilon \subset \mathbb{R}$ for any $\varepsilon > 0$, such that $\vec{u}^{-1}(B_{3\varepsilon}(\pm Q)) = I_\varepsilon$. Then I_ε is decreasing as $\varepsilon \rightarrow +0$, and hence by continuity of \vec{u} , there exists $t \in \mathbb{R}$ such that $\vec{u}(t) = \vec{Q}$, which implies $u \equiv Q$, a contradiction.

Next, any solution u in $\widetilde{\mathcal{T}}_+^0$ corresponds to the case $\gamma_\infty \equiv 0$ in Proposition 3.1, and it is therefore parametrized by $\lambda(0) \in \mathbb{R}$. Since $\lambda(t) \rightarrow 0$ as $t \rightarrow \infty$, the uniqueness in the proposition implies that there are only three solutions converging to Q , modulo time translation: one with $\lambda > 0$ decreasing (for large t), another with

$\lambda < 0$ increasing, and yet another with $\lambda \equiv 0$, i.e., Q . Let W_{\pm} be the two solutions given by $\lambda(0) = \pm\nu/10$ and $\gamma_{\infty} \equiv 0$. Since $E(W_{\pm}) = J(Q)$ and $W_{\pm} \neq Q$, the sign of K_0 is fixed on their trajectories. Indeed, Lemma 4.2 applies to them backward from a neighborhood of $t = \infty$, from which we deduce that $\mathfrak{S}(W_{\pm}(t)) = \mp 1$ away from $t = \infty$, and so $W_+ \in \widetilde{\mathcal{B}}_-^0$ and $W_- \in \widetilde{\mathcal{S}}_-^0$. To prove that W_{\pm} approach Q exponentially, we return to (3.5) and (3.25) with $\gamma_{\infty} = 0$, estimating

$$\|(\lambda, \gamma)\|_{X_T^{k'}} := \|e^{k' \min(t-T, 0)} \lambda(t)\|_{L^1 \cap L^\infty(0, \infty)} + \|e^{k' \min(t-T, 0)} \gamma(t)\|_{St(0, \infty)} \quad (7.26)$$

for $T \rightarrow \infty$ and any fixed $k' \in (0, k)$. The same argument as in the proof of Proposition 3.1 yields

$$\begin{aligned} \|(\lambda, \gamma)\|_{X_T^{k'}} &\lesssim e^{-k'T} [|\lambda(0)| + \|(\lambda, \gamma)\|_X^2] + \|(\lambda, \gamma)\|_X \|(\lambda, \gamma)\|_{X_T^{k'}} \\ &\lesssim e^{-k'T} \nu + \nu \|(\lambda, \gamma)\|_{X_T^{k'}}, \end{aligned} \quad (7.27)$$

and so $\|(\lambda, \gamma)\|_{X_T^{k'}} \lesssim e^{-k'T}$, as desired. \square

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APPENDIX A. TABLE OF NOTATION

symbols	description	defined in
$\mathcal{H}, \mathcal{H}^\varepsilon, \mathcal{H}_{(\delta)}$	energy space and subsets	(1.3), (1.9), (4.35)
$E(\vec{u}), J(u)$	dynamic/static energy	(1.2), (1.6)
$K_s(u), G_s(u)$	scaling derivatives of $J(u)$	(2.3), (2.5)
$N(v), N_\rho(v), N_c(v)$	higher order nonlinearity around Q	(2.12), (3.2)
$C(v)$	higher terms in energy around Q	(4.3)
$V_w(t)$	localized virial	(4.52)
$\ \vec{v}\ _E$	linearized energy norm	(3.7)
$d_Q(\vec{u})$	nonlinear distance to $\pm Q$	(4.8)
$\mathfrak{S}(\vec{u})$	sign functional on $\mathcal{H}_{(\delta)}$	Lemma 4.4
L_+, ω	linearized operator around Q	(1.10), (3.2)
ρ, k, P^+	ground state of L_+	(1.10), (2.20)
$v, \lambda, \gamma, \lambda_{\pm}$	components of u around $\pm Q$	(4.2), (4.9)
δ_E	radius of energy nonlinearity	(4.4)
δ_X	terminal radius of ejection	Lemma 4.2
C_*	constant in a bound on $ K_s $	Lemma 4.2
$\varepsilon_0(\delta), \kappa_0, \kappa_1(\delta)$	constants for variational estimates	Lemma 4.3
δ_S	radius of variational estimates	Lemma 4.4
M_*	energy bound in $\mathfrak{S} = +1$	Lemma 4.5
ε_*	energy threshold for one-pass theorem	(4.46)
δ_*	hyperbolic-variational threshold radius	(4.46)
R_*	exiting radius for one-pass theorem	(4.46)

Relation of the constants:

$$\begin{aligned} 2C_*\delta_S &= \delta_X \leq \delta_E \ll 1 \leq C_*, \quad \mu_0(M_*) > \mu \ll J(Q)^3, \\ \varepsilon_* &= \varepsilon_0(\delta_S) < R_*/2 \ll \min(\delta_S, \kappa_1(\delta_S), \kappa_0\mu^{1/2}, J(Q)^{1/2}). \end{aligned} \quad (\text{A.1})$$

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